The zeros of a polynomial function \( P(x) \) are the same as the roots of the polynomial equation \( P(x) = 0 \). Remember that one of our main goals in algebra is to keep expanding our knowledge of solving equations. In this section we will learn several facts that are useful in solving polynomial equations.

### The Number of Roots to a Polynomial Equation

In solving a polynomial equation by factoring, we find that a factor may occur more than once.

#### Multiplicity

If the factor \( x - c \) occurs \( n \) times in the complete factorization of the polynomial \( P(x) \), then we say that \( c \) is a root of the equation \( P(x) = 0 \) with multiplicity \( n \).

For example, the equation \( x^2 - 10x + 25 = 0 \) is equivalent to \( (x - 5)(x - 5) = 0 \). The only root to this equation is 5. Since the factor \( x - 5 \) occurs twice, we say that 5 is a root with multiplicity 2. Counting multiplicity, every quadratic equation has two roots.

Consider a polynomial equation \( P(x) = 0 \) of positive degree \( n \). By the fundamental theorem of algebra there is at least one complex root \( c_1 \) to this equation. By the factor theorem \( P(x) = 0 \) is equivalent to \( (x - c_1)Q_1(x) = 0 \), where \( Q_1(x) \) is a polynomial with degree \( n - 1 \) (the quotient when \( P(x) \) is divided by \( x - c_1 \)). By the fundamental theorem of algebra there is at least one complex root \( c_2 \) to \( Q_1(x) = 0 \). By the factor theorem \( P(x) = 0 \) can be written as \( (x - c_1)(x - c_2)Q_2(x) = 0 \), where \( Q_2(x) \) is a polynomial with degree \( n - 2 \). Continuing this reasoning \( n \) times, we get a quotient polynomial that has 0 degree, \( n \) factors for \( P(x) \), and \( n \) complex roots, not necessarily all different. We have just proved the following theorem.

#### n-Root Theorem

If \( P(x) = 0 \) is a polynomial equation with real or complex coefficients and positive degree \( n \), then counting multiplicities, \( P(x) = 0 \) has \( n \) roots.

Note that the \( n \)-root theorem also means that a polynomial function of positive degree \( n \) has \( n \) zeros, counting multiplicities.

### Example 1

**Finding all roots to a polynomial equation**

State the degree of each polynomial equation. Find all of the real and imaginary roots to each equation, stating multiplicity when it is greater than 1.

**a)** \( 6x^5 + 24x^3 = 0 \)  
**b)** \( (x - 3)^2(x + 4)^3 = 0 \)

**Solution**

**a)** The equation is a fifth-degree equation. We can solve it by factoring:

\[
6x^3(x^2 + 4) = 0
\]

\[
6x^3 = 0 \quad \text{or} \quad x^2 + 4 = 0
\]

\[
x^3 = 0 \quad \text{or} \quad x^2 = -4
\]

\[
x = 0 \quad \text{or} \quad x = \pm 2i
\]
The roots are $\pm 2i$ and 0. Since 0 is a root with multiplicity 3, counting multiplicities there are five roots.

b) If we would multiply the factors in this polynomial equation, then the highest power of $x$ would be 7. So the degree of the equation is 7. There are only two distinct roots to the equation, 3 and $-4$. We say that 3 is a root with multiplicity 2 and $-4$ is a root with multiplicity 5. So counting multiplicities, there are seven roots to the equation.

**The Conjugate Pairs Theorem**

The solutions to the quadratic equation $x^2 - 2x + 5 = 0$ are the complex numbers $1 - 2i$ and $1 + 2i$. These numbers are conjugates of one another. The quadratic formula guarantees that complex solutions of quadratic equations with real coefficients occur in conjugate pairs. This situation also occurs for polynomial equations of higher degree.

**Conjugate Pairs Theorem**

If $P(x) = 0$ is a polynomial equation with real coefficients and the complex number $a + bi$ ($b \neq 0$) is a root, then the complex number $a - bi$ is also a root.

**Finding an equation with given roots**

Find a polynomial equation with real coefficients that has 2 and $1 - i$ as roots.

**Solution**

Since the polynomial is to have real coefficients, the imaginary roots occur in conjugate pairs. So a polynomial with these two roots actually must have at least three roots: 2, $1 - i$, and $1 + i$. Since each root of the equation comes from a factor of the polynomial, we can write the following equation:

$$\begin{align*}
(x - 2)(x - [1 - i])(x - [1 + i]) &= 0 \\
(x - 2)(x^2 - 2x + 2) &= 0 \\
x^3 - 4x^2 + 6x - 4 &= 0
\end{align*}$$

This equation has the required solutions and the smallest degree. Any multiple of this equation would also have the required solutions but would not be as simple.

**Descartes’ Rule of Signs**

None of the theorems in this chapter tells us how to find all of the $n$ roots to a polynomial equation of degree $n$. The theorems and rules presented here add to our knowledge of polynomial equations and help us to solve more equations. Descartes’ rule of signs is a method for looking at a polynomial equation and estimating the number of positive, negative, and imaginary solutions.

When a polynomial is written in descending order, a variation of sign occurs when the signs of consecutive terms change. For example, if $P(x) = 3x^3 - 7x^4 - 8x^3 - x^2 + 3x - 9$, there are sign changes going from the first to the second terms, from the fourth to the fifth terms, and from the fifth to the sixth terms. So
there are three variations in sign for \( P(x) \). Descartes’ rule requires that we also count the variations in sign for \( P(-x) \) after it is simplified:

\[
P(-x) = 3(-x)^5 - 7(-x)^4 - 8(-x)^3 - (-x)^2 + 3(-x) - 9
\]
\[
= -3x^5 - 7x^4 + 8x^3 - x^2 - 3x - 9
\]

In the polynomial \( P(-x) \) the signs of the terms change from the second to the third terms, and then the signs change again from the third to the fourth terms. So there are two variations in sign for \( P(-x) \).

**Descartes’ Rule of Signs**

If \( P(x) = 0 \) is a polynomial equation with real coefficients, then the number of positive roots of the equation is either equal to the number of variations of sign of \( P(x) \) or less than that by an even number. The number of negative roots of the equation is either equal to the number of variations in sign of \( P(-x) \) or less than that by an even number.

**Example 3**

Discuss the possibilities for the roots to \( 2x^3 - 5x^2 - 6x + 4 = 0 \).

**Solution**

The number of variations of sign in \( P(x) = 2x^3 - 5x^2 - 6x + 4 \) is 2. According to Descartes’ rule, the number of positive roots is either 2 or 0. Since \( P(-x) = 2(-x)^3 - 5(-x)^2 - 6(-x) + 4 = -2x^3 - 5x^2 + 6x + 4 \), there is only one variation of sign in \( P(-x) \). So there is exactly one negative root. If only one negative root exists, then the other two roots must be positive or imaginary. The number of imaginary roots is determined by the number of positive and negative roots because the total number of roots must be three. The following table summarizes these two possibilities.

<table>
<thead>
<tr>
<th>Number of Positive Roots</th>
<th>Number of Negative Roots</th>
<th>Number of Imaginary Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

**Example 4**

Discuss the possibilities for the roots to \( 3x^4 - 5x^3 - x^2 - 8x + 4 = 0 \).

**Solution**

The number of variations of sign in \( P(x) = 3x^4 - 5x^3 - x^2 - 8x + 4 \) is two. According to Descartes’ rule, there are either two or no positive roots to the equation. Since \( P(-x) = 3(-x)^4 - 5(-x)^3 - (-x)^2 - 8(-x) + 4 = 3x^4 + 5x^3 - x^2 + 8x + 4 \), there are two variations of sign in \( P(-x) \). So the number of negative roots is either two or zero. Each line of the following table gives a possible distribution of the type of roots to the equation.
Descartes’ rule of signs adds to our knowledge of the roots of an equation. It is especially helpful when the number of variations of sign is zero or one. If there are no variations in sign for $P(x)$, then there are no positive roots. If there is one variation of sign in $P(x)$, then we know that one positive root exists.

### Bounds on the Roots

The next theorem on roots has to do with determining the size of the roots.

#### Theorem on Bounds

Suppose $P(x)$ is a polynomial with real coefficients and a positive leading coefficient, and synthetic division with $c$ is performed.

- If $c > 0$ and all terms in the bottom row are nonnegative, then no number greater than $c$ can be a root of $P(x) = 0$.
- If $c < 0$ and the terms in the bottom row alternate in sign, then no number less than $c$ can be a root of $P(x) = 0$.

If there are no roots greater than $c$, then $c$ is called an **upper bound** for the roots. If there are no roots less than $c$, then $c$ is called a **lower bound** for the roots. If 0 appears in the bottom row of the synthetic division, we may consider it as positive or negative term in determining whether the signs alternate.

### Example 5

**Integral bounds for the roots**

Use the theorem on bounds to establish the best integral bounds for the roots of $2x^3 - 5x^2 - 6x + 4 = 0$.

#### Solution

Try synthetic division with the integers 1, 2, 3, and so on. The first integer for which all terms on the bottom row are nonnegative is the best upper bound for the roots:

```
  1 |  2  -5  -6   4  
     |      2   -3   -9  
     |__________  
     |  2  -3   -9   -5

  2 |  2  -5  -6   4  
     |      4   -2   -16  
     |__________  
     |  2  -1   -8   -12

  3 |  2  -5  -6   4  
     |      6    3   -9  
     |__________  
     |  2   1   -3   -5

  4 |  2  -5  -6   4  
     |      8    12   24  
     |__________  
     |  2   3    6   28
```

By the theorem on bounds no number greater than 4 can be a root to the equation. Now try synthetic division with the integers $-1, -2, -3$, and so on. The first
negative integer for which the terms on the bottom row alternate in sign is the best lower bound for the roots:

\[
\begin{array}{c|cccc}
-1 & 2 & -5 & -6 & 4 \\
-2 & 7 & -1 & & \\
\hline
 & 2 & -7 & 1 & 3 \\
\end{array}
\quad \begin{array}{c|cccc}
-2 & 2 & -5 & -6 & 4 \\
-2 & 14 & -16 & & \\
\hline
 & 2 & -7 & 8 & -12 \\
\end{array}
\]

By the theorem on bounds no number less than \(-2\) can be a root to the equation. So all of the real roots to this equation are between \(-2\) and 4.

In the next example we will use all of the information available to find all of the solutions to a polynomial equation.

**Example 6**

**Finding all solutions to a polynomial equation**

Find all of the solutions to \(2x^3 - 5x^2 - 6x + 4 = 0\).

**Solution**

In Example 3 we saw that this equation has either two positive roots and one negative root or one negative root and two imaginary roots. In Example 5 we saw that all of the real roots to this equation are between \(-2\) and 4. From the rational root theorem we have \(-1, -\frac{1}{2}, \frac{1}{2}, 4\) as the possible rational roots. Since there must be one negative root and it must be greater than \(-2\), the only possible numbers from the list are \(-1\) and \(-\frac{1}{2}\). So start by checking \(-\frac{1}{2}\) and \(-1\) with synthetic division:

\[
\begin{array}{c|ccccc}
-\frac{1}{2} & 2 & -5 & -6 & 4 & \frac{3}{2} \\
& -1 & 3 & \frac{3}{2} & & \\
\hline
& 2 & -6 & -3 & \frac{11}{2} & \\
\end{array}
\quad \begin{array}{c|ccccc}
-1 & 2 & -5 & -6 & 4 & \\
& -2 & 7 & -1 & & \\
\hline
& 2 & -7 & 1 & 3 & \\
\end{array}
\]

Since neither \(-1\) nor \(-\frac{1}{2}\) is a root, the negative root must be irrational. Since there might be two positive roots smaller than 4, we check \(\frac{1}{2}, 1, \) and 2:

\[
\begin{array}{c|ccccc}
\frac{1}{2} & 2 & -5 & -6 & 4 & \frac{3}{2} \\
& 1 & -2 & -4 & & \\
\hline
& 2 & -4 & -8 & 0 & \\
\end{array}
\]

Since \(\frac{1}{2}\) is a root of the equation, \(x - \frac{1}{2}\) is a factor of the polynomial:

\[
\left(x - \frac{1}{2}\right)(2x^2 - 4x - 8) = 0
\]

\[
(2x - 1)(x^2 - 2x - 4) = 0
\]

\[2x - 1 = 0 \quad \text{or} \quad x^2 - 2x - 4 = 0\]

\[x = \frac{1}{2} \quad \text{or} \quad x = \frac{2 \pm \sqrt{4 - 4(1)(-4)}}{2} = 1 \pm \sqrt{5}\]

There are two positive roots, \(\frac{1}{2}\) and \(1 + \sqrt{5}\). The negative root is \(1 - \sqrt{5}\). Note that the roots guaranteed by Descartes’ rule of signs are real numbers but not necessarily rational numbers.
WARM-UPS

True or false? Explain your answer.
1. The number 3 is a root of \( x^2 - 9 = 0 \) with multiplicity 2.
2. Counting multiplicities, the equation \( x^8 = 1 \) has eight solutions in the set of complex numbers.
3. The number \( \frac{2}{3} \) is a root of multiplicity 4 for the equation \((3x - 2)^4(x^2 + 2x + 1) = 0\).
4. The number 2 is a root of multiplicity 3 for the equation \((x - 2)^3(x^2 + x - 1) = 0\).
5. If \( 2 - 3i \) is a solution to a polynomial equation with real coefficients, then \(-2 + 3i\) is also a solution to the equation.
6. If \( P(x) = 0 \) is a polynomial equation with real coefficients and \( 5 - 4i \) and \( 3 + 6i \) are solutions to \( P(x) = 0 \), then the degree of \( P(x) \) is at least 4.
7. Both \(-1 - i\sqrt{2}\) and \(1 - i\sqrt{2}\) are solutions to \( 7x^3 - 5x^2 + 6x - 8 = 0 \).
8. If \( P(x) = x^3 - 6x^2 + 3x - 2 \), then \( P(-x) = -x^3 + 6x^2 - 3x + 2\).
9. The equation \( x^3 + 5x^2 + 6x + 1 = 0 \) has no positive solutions.
10. The equation \( x^3 - 5 = 0 \) has two imaginary solutions.

10.2 EXERCISES

Reading and Writing  After reading this section, write out the answers to these questions. Use complete sentences.
1. What is multiplicity?
2. What is the n-root theorem?
3. What is the conjugate pairs theorem?
4. What is Descartes’ rule of signs?
5. What is an upper bound for the roots of a polynomial?
6. What is a lower bound for the roots?

State the degree of each polynomial equation. Find all of the real and imaginary roots to each equation. State the multiplicity of a root when it is greater than 1. See Example 1.
7. \( x^5 - 4x^3 = 0 \)
8. \( x^6 + 9x^4 = 0 \)
9. \( x^4 + 2x^3 + x^2 = 0 \)
10. \( x^5 - 4x^4 + 4x^3 = 0 \)
11. \( x^4 - 6x^2 + 9 = 0 \)
12. \( x^4 - 8x^2 + 16 = 0 \)
13. \( (x - 1)^3(x + 2)^2 = 0 \)
14. \( (2x + 1)^2(3x - 5)^4 = 0 \)
15. \( x^4 - 2x^2 + 1 = 0 \)
16. \( 4x^4 - 4x^2 + 1 = 0 \)
Find a polynomial equation with real coefficients that has the
given roots. See Example 2.
17. 3, 2 − i
18. −4, 3 + i
19. −2, i
20. 4, −i
21. 0, i√2
22. −3, i√3
23. i, 1 − i
24. 2i, −i
25. 1, 2
26. 1
27. −1, 2, 3
28. −2, 3, 2

Discuss the possibilities for the roots to each equation. Do not
solve the equation. See Examples 3 and 4.
29. x^3 + 3x^2 + 5x + 7 = 0
30. 2x^3 − 3x^2 + 5x − 6 = 0
31. −2x^3 − x^2 + 3x + 2 = 0
32. x^3 + x^2 − 5x − 1 = 0
33. x^3 + x^3 + 3x^2 + 2x + 6 = 0
34. x^4 − 1 = 0
35. x^3 + x^2 + 1 = 0
36. x^6 + 3x^4 + 2x^2 + 6 = 0
37. x^3 + x − 1 = 0
38. −x^3 + 3x^3 + 5x + 5 = 0
39. x^5 + x^3 + 3x = 0
40. x^3 − 5x^2 + 6x = 0

Establish the best integral bounds for the roots of each equation
according to the theorem on bounds. See Example 5.
41. x^4 − 5x^3 + 7 = 0
42. 2x^3 − x^2 − 7x + 7 = 0
43. 2x^3 − 5x^2 + 9x − 18 = 0
44. x^2 − 7x − 16 = 0
45. x^2 + x − 13 = 0
46. x^3 − 15x + 25 = 0
47. 2x^3 − 13x^2 + 25x − 14 = 0
48. x^3 − 6x^2 + 11x − 6 = 0

Use the rational root theorem, Descartes’ rule of signs, and
the theorem on bounds as aids in finding all solutions to each
equation. See Example 6.
49. x^3 + x + 10 = 0
50. x^3 − 7x^2 + 17x − 15 = 0
51. 2x^3 − 5x^2 − 6x + 4 = 0
52. 3x^3 − 17x^2 + 12x + 6 = 0
53. 4x^3 − 6x^2 − 2x + 1 = 0
54. x^3 + 5x^2 − 20x − 42 = 0
55. x^4 − 5x^3 + 5x^2 + 5x − 6 = 0
56. x^4 − 2x^3 + 5x^2 − 8x + 4 = 0
57. x^4 − 7x^3 + 17x^2 − 17x + 6 = 0
58. x^4 + 7x^3 + 17x^2 + 17x + 6 = 0
59. x^6 − x^5 + 2x^3 − 2x^2 − 15x + 15 = 0
60. 2x^6 + 4x^5 + x^4 + 2x^3 − x^2 − 2x = 0

Solve each problem.
61. Willard is designing a cylindrical tank with cone-shaped
ends. The length of the cylinder is to be 20 feet (ft) larger
than the radius of the cylinder, and the height of the cone is
2 ft. If the volume of the tank is 984π cubic feet (ft^3), then
what is the radius of the cylinder?

![Figure for Exercise 61](image)

62. Dr. Hu is designing a chemical storage tank in the shape of
a cylinder with hemispherical ends. If the length of the
cylinder is to be 20 ft larger than its radius and the volume is
to be 3,321π ft^3, then what is the radius?

![Figure for Exercise 62](image)
63. A box of frozen specimens measures 4 inches by 5 inches by 3 inches. It is wrapped in an insulating material of uniform thickness for shipment. The volume of the box including the insulating material is 120 cubic inches (in.$^3$). How thick is the insulation?

64. An independent marketing research agency has determined that the best box for breakfast cereal has a height that is 6 inches (in.) larger than its thickness and a width that is 5 in. larger than its thickness. If such a box is to have a volume of 112 in.$^3$, then what should the thickness be?

**Graphing Calculator Exercises**

Find all real roots to each polynomial equation by graphing the corresponding function and locating the x-intercepts.

65. $x^4 - 12x^2 + 10 = 0$

66. $x^5 + x^4 - 7x^3 - 7x^2 + 12x + 12 = 0$

67. $x^6 - 9x^4 + 20x^2 - 12 = 0$

68. $4x^5 + 16x^4 - 5x^3 - 20x^2 + x + 8 = 0$

# 10.3 Graphs of Polynomial Functions

In Chapter 3 we learned that the graph of a polynomial function of degree 0 or 1 is a straight line and that the graph of a second-degree polynomial function is a parabola. In this section we will concentrate on graphs of polynomial functions of degree larger than 2.

**Symmetry**

Consider the graph of the quadratic function $f(x) = x^2$ shown in Fig. 10.1. Notice that both (2, 4) and $(-2, 4)$ are on the graph. In fact, $f(x) = f(-x)$ for any value of $x$. We get the same $y$-coordinate whether we evaluate the function at a number or its opposite. This fact causes the graph to be symmetric about the y-axis. If we folded the paper along the y-axis, the two halves of the graph would coincide.

**Symmetric about the y-Axis**

If $f(x)$ is a function such that $f(x) = f(-x)$ for any value of $x$ in its domain, then the graph of the function is said to be symmetric about the y-axis.

Consider the graph of $f(x) = x^3$ shown in Fig. 10.2. It is not symmetric about the y-axis like the graph of $f(x) = x^2$, but it has a different kind of symmetry. On the graph of $f(x) = x^3$ we find the points (2, 8) and (-2, -8). In this case $f(x)$ and $f(-x)$ are not equal, but $f(-x) = -f(x)$. Notice that the points (2, 8) and (-2, -8) are the same distance from the origin and lie on a line through the origin.

**Symmetric about the Origin**

If $f(x)$ is a function such that $f(-x) = -f(x)$ for any value of $x$ in its domain, then the graph of the function is said to be symmetric about the origin.