The nineteenth-century physician and physicist, Jean Louis Marie Poiseuille (1799–1869) is given credit for discovering a formula associated with the circulation of blood through arteries. Poiseuille’s law, as it is known, can be used to determine the velocity of blood in an artery at a given distance from the center of the artery. The formula states that the flow of blood in an artery is faster toward the center of the blood vessel and is slower toward the outside. Blood flow can also be affected by a person’s blood pressure, the length of the blood vessel, and the viscosity of the blood itself.

In later years, Poiseuille’s continued interest in blood circulation led him to experiments to show that blood pressure rises and falls when a person exhales and inhales. In modern medicine, physicians can use Poiseuille’s law to determine how much the radius of a blocked blood vessel must be widened to create a healthy flow of blood.

In this chapter you will study polynomials, the fundamental expressions of algebra. Polynomials are to algebra what integers are to arithmetic. We use polynomials to represent quantities in general, such as perimeter, area, revenue, and the volume of blood flowing through an artery. In Exercise 85 of Section 4.4, you will see Poiseuille’s law represented by a polynomial.
We first used polynomials in Chapter 1 but did not identify them as polynomials. Polynomials also occurred in the equations and inequalities of Chapter 2. In this section we will define polynomials and begin a thorough study of polynomials.

**Polynomials**

In Chapter 1 we defined a **term** as an expression containing a number or the product of a number and one or more variables raised to powers. Some examples of terms are $4x^3$, $-x^2y^3$, $6ab$, and $-2$.

A **polynomial** is a single term or a finite sum of terms. The powers of the variables in a polynomial must be positive integers. For example,

$$4x^3 + (-15x^2) + x + (-2)$$

is a polynomial. Because it is simpler to write addition of a negative as subtraction, this polynomial is usually written as

$$4x^3 - 15x^2 + x - 2.$$  

The **degree of a polynomial** in one variable is the highest power of the variable in the polynomial. So $4x^3 - 15x^2 + x - 2$ has degree 3 and $7w - w^2$ has degree 2.

The **degree of a term** is the power of the variable in the term. Because the last term has no variable, its degree is 0.

<table>
<thead>
<tr>
<th>Degree</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$4x^3$</td>
</tr>
<tr>
<td>2</td>
<td>$-15x^2$</td>
</tr>
<tr>
<td>1</td>
<td>$x$</td>
</tr>
<tr>
<td>0</td>
<td>Constant</td>
</tr>
</tbody>
</table>

A single number is called a **constant** and so the last term is the **constant term**. The degree of a polynomial consisting of a single number such as 8 is 0.

The number preceding the variable in each term is called the **coefficient** of that variable or the coefficient of that term. In $4x^3 - 15x^2 + x - 2$ the coefficient of $x^3$ is 4, the coefficient of $x^2$ is $-15$, and the coefficient of $x$ is 1 because $x = 1 \cdot x$.

**Example 1**

**Identifying coefficients**

Determine the coefficients of $x^3$ and $x^2$ in each polynomial:

a) $x^3 + 5x^2 - 6$  

b) $4x^6 - x^3 + x$

**Solution**

a) Write the polynomial as $1 \cdot x^3 + 5x^2 - 6$ to see that the coefficient of $x^3$ is 1 and the coefficient of $x^2$ is 5.

b) The $x^2$-term is missing in $4x^6 - x^3 + x$. Because $4x^6 - x^3 + x$ can be written as $4x^6 - 1 \cdot x^3 + 0 \cdot x^2 + x$,

the coefficient of $x^3$ is $-1$ and the coefficient of $x^2$ is 0.

For simplicity we generally write polynomials with the exponents decreasing from left to right and the constant term last. So we write

$$x^3 - 4x^2 + 5x + 1$$

rather than   

$$-4x^2 + 1 + 5x + x^3.$$  

When a polynomial is written with decreasing exponents, the coefficient of the first term is called the **leading coefficient**.
Certain polynomials are given special names. A **monomial** is a polynomial that has one term, a **binomial** is a polynomial that has two terms, and a **trinomial** is a polynomial that has three terms. For example, $3x^3$ is a monomial, $2x - 1$ is a binomial, and $4x^6 - 3x + 2$ is a trinomial.

### Example 2

**Types of polynomials**

Identify each polynomial as a monomial, binomial, or trinomial and state its degree.

- **a)** $5x^2 - 7x^3 + 2$
- **b)** $x^{43} - x^2$
- **c)** $5x$
- **d)** $-12$

**Solution**

- **a)** The polynomial $5x^2 - 7x^3 + 2$ is a third-degree trinomial.
- **b)** The polynomial $x^{43} - x^2$ is a binomial with degree $43$.
- **c)** Because $5x = 5x^1$, this polynomial is a monomial with degree $1$.
- **d)** The polynomial $-12$ is a monomial with degree $0$.

### Example 3

**Value of a Polynomial**

A polynomial is an algebraic expression. Like other algebraic expressions involving variables, a polynomial has no specific value unless the variables are replaced by numbers. A polynomial can be evaluated with or without the function notation discussed in Chapter 3.

**Evaluating polynomials**

- **a)** Find the value of $-3x^4 - x^3 + 20x + 3$ when $x = 1$.
- **b)** Find the value of $-3x^4 - x^3 + 20x + 3$ when $x = -2$.
- **c)** If $P(x) = -3x^4 - x^3 + 20x + 3$, find $P(1)$.

**Solution**

- **a)** Replace $x$ by $1$ in the polynomial:

$$-3x^4 - x^3 + 20x + 3 = -3(1)^4 - (1)^3 + 20(1) + 3$$

$$= -3 - 1 + 20 + 3$$

$$= 19$$

So the value of the polynomial is $19$ when $x = 1$.

- **b)** Replace $x$ by $-2$ in the polynomial:

$$-3x^4 - x^3 + 20x + 3 = -3(-2)^4 - (-2)^3 + 20(-2) + 3$$

$$= -3(16) - (-8) - 40 + 3$$

$$= -48 + 8 - 40 + 3$$

$$= -77$$

So the value of the polynomial is $-77$ when $x = -2$.

- **c)** This is a repeat of part (a) using the function notation from Chapter 3. $P(1)$, read “$P$ of $1$,” is the value of the polynomial $P(x)$ when $x$ is $1$. To find $P(1)$, replace $x$ by $1$ in the formula for $P(x)$:

$$P(x) = -3x^4 - x^3 + 20x + 3$$

$$P(1) = -3(1)^4 - (1)^3 + 20(1) + 3$$

$$= -3 - 1 + 20 + 3$$

$$= 19$$

So $P(1) = 19$. The value of the polynomial when $x = 1$ is $19$. 

---

**Study Tip**

Be active in class. Don’t be embarrassed to ask questions or answer questions. You can often learn more from a wrong answer than a right one. Your instructor knows that you are not yet an expert in algebra. Instructors love active classrooms and they will not think less of you for speaking out.
**Addition of Polynomials**

You learned how to combine like terms in Chapter 1. Also, you combined like terms when solving equations in Chapter 2. Addition of polynomials is done simply by adding the like terms.

**Addition of Polynomials**

To add two polynomials, add the like terms.

Polynomials can be added horizontally or vertically, as shown in the next example.

**Example 4**

**Adding polynomials**

Perform the indicated operation.

a) \((x^2 - 6x + 5) + (-3x^2 + 5x - 9)\)

b) \((-5a^3 + 3a - 7) + (4a^2 - 3a + 7)\)

**Solution**

a) We can use the commutative and associative properties to get the like terms next to each other and then combine them:

\[
(x^2 - 6x + 5) + (-3x^2 + 5x - 9) = x^2 - 3x^2 - 6x + 5x + 5 - 9 = -2x^2 - x - 4
\]

b) When adding vertically, we line up the like terms:

\[
\begin{array}{c}
-5a^3 \\
4a^2 \\
\hline
-5a^3 + 4a^2 - 3a + 7
\end{array}
\]

Add.

**Subtraction of Polynomials**

When we subtract polynomials, we subtract the like terms. Because \(a - b = a + (-b)\), we can subtract by adding the opposite of the second polynomial to the first polynomial. Remember that a negative sign in front of parentheses changes the sign of each term in the parentheses. For example,

\[-(x^2 - 2x + 8) = -x^2 + 2x - 8.\]

Polynomials can be subtracted horizontally or vertically, as shown in the next example.

**Example 5**

**Subtracting polynomials**

Perform the indicated operation.

a) \((x^2 - 5x - 3) - (4x^2 + 8x - 9)\)

b) \((4y^3 - 3y + 2) - (5y^2 - 7y - 6)\)

**Solution**

a) \((x^2 - 5x - 3) - (4x^2 + 8x - 9) = x^2 - 5x - 3 - 4x^2 - 8x + 9\)

\[= -3x^2 - 13x + 6\]

Change signs.

Add.
b) To subtract $5y^2 - 7y - 6$ from $4y^3 - 3y + 2$ vertically, we line up the like terms as we do for addition:

$$
4y^3 - 3y + 2 \\
\underline{-(5y^2 - 7y - 6)}
$$

Now change the signs of $5y^2 - 7y - 6$ and add the like terms:

$$
4y^3 - 3y + 2 \\
\underline{+5y^2 - 7y - 6} \\
4y^3 + 5y^2 - 4y + 8
$$

\textbf{CAUTION} When adding or subtracting polynomials vertically, be sure to line up the like terms.

In the next example we combine addition and subtraction of polynomials.

\textbf{Example 6}

Adding and subtracting

Perform the indicated operations:

$$
(2x^2 - 3x) + (x^3 + 6) - (x^4 - 6x^2 - 9)
$$

\textbf{Solution}

Remove the parentheses and combine the like terms:

$$
(2x^2 - 3x) + (x^3 + 6) - (x^4 - 6x^2 - 9) = 2x^2 - 3x + x^3 + 6 - x^4 + 6x^2 + 9 \\
= -x^4 + x^3 + 8x^2 - 3x + 15
$$

\textbf{Applications}

Polynomials are often used to represent unknown quantities. In certain situations it is necessary to add or subtract such polynomials.

\textbf{Example 7}

Profit from prints

Trey pays $60 per day for a permit to sell famous art prints in the Student Union Mall. Each print costs him $4, so the polynomial $4x + 60$ represents his daily cost in dollars for $x$ prints sold. He sells the prints for $10 each. So the polynomial $10x$ represents his daily revenue for $x$ prints sold. Find a polynomial that represents his daily profit from selling $x$ prints. Evaluate the profit polynomial for $x = 30$.

\textbf{Solution}

Because profit is revenue minus cost, we can subtract the corresponding polynomials to get a polynomial that represents the daily profit:

$$
10x - (4x + 60) = 10x - 4x - 60 \\
= 6x - 60
$$

His daily profit from selling $x$ prints is $6x - 60$ dollars. Evaluate this profit polynomial for $x = 30$:

$$
6x - 60 = 6(30) - 60 = 120
$$

So if Trey sells 30 prints, his profit is $120.
The message we hear today about healthy eating is “more fiber and less fat." But healthy eating can be challenging. Jill Brown, Registered Dietitian and owner of Healthy Habits Nutritional Counseling and Consulting, provides nutritional counseling for people who are basically healthy but interested in improving their eating habits.

On a client’s first visit, an extensive nutrition, medical, and family history is taken. Behavior patterns are discussed, and nutritional goals are set. Ms. Brown strives for a realistic rather than an idealistic approach. Whenever possible, a client should eat foods that are high in vitamin content rather than take vitamin pills. Moreover, certain frame types and family history make a slight and slender look difficult to achieve. Here, Ms. Brown might use the Harris-Benedict formula to determine the daily basal energy expenditure based on age, gender, and size. In addition to diet, exercise is discussed and encouraged. Finally, Ms. Brown provides a support system and serves as a “nutritional coach.” By following her advice, many of her clients lower their blood pressure, reduce their cholesterol, have more energy, and look and feel better.

In Exercises 101 and 102 of this section you will use the Harris-Benedict formula to calculate the basal energy expenditure for a female and a male.

**WARM-UPS**

**True or false? Explain your answer.**

1. In the polynomial $2x^2 - 4x + 7$ the coefficient of $x$ is 4. 
2. The degree of the polynomial $x^2 + 5x - 9x^3 + 6$ is 2. 
3. In the polynomial $x^2 - x$ the coefficient of $x$ is $-1$. 
4. The degree of the polynomial $x^2 - x$ is 2. 
5. A binomial always has a degree of 2. 
6. The polynomial $3x^2 - 5x + 9$ is a trinomial. 
7. Every trinomial has degree 2. 
8. $x^2 - 7x^2 = -6x^2$ for any value of $x$. 
9. $(3x^2 - 8x + 6) + (x^2 + 4x - 9) = 4x^2 - 4x - 3$ for any value of $x$. 
10. $(x^2 - 4x) - (x^2 - 3x) = -7x$ for any value of $x$. 

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a term? 
2. What is a polynomial? 
3. What is the degree of a polynomial?
4. What is the value of a polynomial?

5. How do we add polynomials?

6. How do we subtract polynomials?

Determine the coefficients of $x^3$ and $x^2$ in each polynomial. See Example 1.

7. $-3x^3 + 7x^2$
8. $10x^3 - x^2$
9. $x^3 + 6x^2 - 9$
10. $x^3 - x^3 + 3$
11. $\frac{x^3}{3} + 7x^2 - 4$
12. $\frac{x^3}{2} - \frac{x^2}{4} + 2x + 1$

Identify each polynomial as a monomial, binomial, or trinomial and state its degree. See Example 2.

13. $-1$
14. $5$
15. $m^3$
16. $3a^8$
17. $4x + 7$
18. $a + 6$
19. $x^{10} - 3x^2 + 2$
20. $y^6 - 6y^3 + 9$
21. $x^6 + 1$
22. $b^2 - 4$
23. $a^3 - a^2 + 5$
24. $-x^2 + 4x - 9$

Evaluate each polynomial as indicated. See Example 3.

25. Evaluate $2x^3 - 3x + 1$ for $x = -1$.
26. Evaluate $3x^2 - x + 2$ for $x = -2$.
27. Evaluate $-3x^3 - x^2 + 3x - 4$ for $x = 3$.
28. Evaluate $-2x^4 - 3x^2 + 5x - 9$ for $x = 2$.
29. If $P(x) = 3x^4 - 2x^3 + 7$, find $P(-2)$.
30. If $P(x) = -2x^3 + 5x^2 - 12$, find $P(5)$.
31. If $P(x) = 1.2x^3 - 4.3x^2 - 2.4$, find $P(1.45)$.
32. If $P(x) = -3.5x^4 - 4.6x^3 + 5.5$, find $P(-2.36)$.

Perform the indicated operation. See Example 4.

33. $(x - 3) + (3x - 5)$
34. $(x - 2) + (x + 3)$
35. $(q - 3) + (q + 3)$
36. $(q + 4) + (q + 6)$
37. $(3x + 2) + (x^2 - 4)$
38. $(5x^2 - 2) + (-3x^2 - 1)$
39. $(4x - 1) + (x^3 + 5x - 6)$
40. $(3x - 7) + (x^2 - 4x + 6)$
41. $(a^2 - 3a + 1) + (2a^2 - 4a - 5)$
42. $(w^2 - 2w + 1) + (2w - 5 + w^2)$
43. $(w^2 - 9w - 3) + (w - 4w^2 + 8)$
44. $(a^3 - a^2 - 5a) + (6 - a - 3a^2)$
45. $(5.76x^2 - 3.14x - 7.09) + (3.9x^3 + 1.21x + 5.6)$

46. $(8.5x^2 + 3.27x - 9.33) + (x^2 - 4.39x - 2.32)$

Perform the indicated operation. See Example 5.
47. $(x - 2) - (5x - 8)$
48. $(x - 7) - (3x - 1)$
49. $(m - 2) - (m + 3)$
50. $(m + 5) - (m + 9)$
51. $(2z^2 - 3z) - (3z^2 - 5z)$
52. $(z^2 - 4z) - (5z^2 - 3z)$
53. $(w^5 - w^3) - (-w^4 + w^3)$
54. $(w^5 - w^3) - (-w^2 + w)$
55. $(t^4 - 3t + 4) - (t^2 - 5t - 9)$
56. $(t^3 - 6t + 7) - (5t^2 - 3t - 2)$
57. $(9 - 3y - 5y^2) - (2 + 5y - y^2)$
58. $(4 - 5y + y^3) - (2 - 3y + y^2)$
59. $(3.55x - 879) - (26.4x - 455.8)$
60. $(345.56x - 347.4) - (56.6x + 433)$

Add or subtract the polynomials as indicated. See Examples 4 and 5.

61. Add:
$$3a - 4$$
$$a + 6$$
$$w + 3$$

62. Add:
$$2w - 8$$

63. Subtract:
$$3x + 11$$
$$4x + 3$$
$$5x + 7$$
$$2x + 9$$

64. Subtract:
$$a + b$$
$$a - b$$
$$s - 1$$

65. Add:
$$s - 6$$

66. Add:
$$a - b$$
$$a + b$$

67. Subtract:
$$-3m + 1$$
$$-5n + 2$$
$$2m - 6$$
$$3n - 4$$

68. Subtract:
$$2x^2 - x - 3$$
$$2x^2 + x + 4$$
$$3x^2 - x - 5$$

69. Add:
$$2x^2 - x - 3$$
$$2x^2 + x + 4$$
$$3x^2 - x - 5$$

70. Add:
$$-x^3 + 4x - 6$$
$$x^3 - 3x + 6$$
$$3x^2 - x - 5$$

71. Subtract:
$$3a^3 - 5a^2 + 7$$
$$2a^3 + 4a^2 - 2a$$

72. Subtract:
$$-2b^3 + 7b^2 - 9$$
$$b^3 - 4b - 2$$

73. Subtract:
$$x^2 - 3x + 6$$
$$x^2 - 3$$
$$3x^4 - x^2$$

74. Subtract:
$$x^2 - 3x^2 + 2$$
$$3x^4 - 2x^2$$

75. Add:
$$y^3 + 4y^2 - 6y - 5$$

76. Add:
$$q^2 - 4q + 9$$
$$y^3 + 3y^2 + 2y - 9$$
$$-3q^2 - 7q + 5$$
Perform the operations indicated.
77. Find the sum of $2m - 9$ and $3m + 4$.
78. Find the sum of $-3n - 2$ and $6m - 3$.
79. Find the difference when $7y - 3$ is subtracted from $9y - 2$.
80. Find the difference when $-2y - 1$ is subtracted from $3y - 4$.
81. Subtract $x^2 - 3x - 1$ from the sum of $2x^2 - x + 3$ and $x^2 + 5x - 9$.
82. Subtract $-2y^2 + 3y - 8$ from the sum of $-3y^2 - 2y + 6$ and $7y^2 + 8y - 3$.

Perform the indicated operations. See Example 6.
83. $(4m - 2) + (2m + 4) - (9m - 1)$
84. $(-5m - 6) + (8m - 3) - (-5m + 3)$
85. $(6y - 2) - (8y + 3) - (9y - 2)$
86. $(-5y - 1) - (8y - 4) - (y + 3)$
87. $(-x^2 - 5x + 4) + (6x^2 - 8x + 9) - (3x^2 - 7x + 1)$
88. $(-8x^2 + 5x - 12) + (-3x^2 - 9x + 18)$
$(-3x^2 + 9x - 4)$
89. $(-6z^4 - 3z^3 + 7z^2) - (5z^3 + 3z^2 - 2)$
$(z^4 - z^2 + 5)$
90. $(-v^3 - v^2 - 1) - (v^4 - v^2 - v - 1) + (v^3 - 3v^2 + 6)$

Solve each problem. See Example 7.
91. Profitable pumps. Walter Waterman, of Walter’s Water Pumps in Winnipeg has found that when he produces $x$ water pumps per month, his revenue is $x^2 + 400x + 300$ dollars. His cost for producing $x$ water pumps per month is $x^2 + 300x - 200$ dollars. Write a polynomial that represents his monthly profit for $x$ water pumps. Evaluate this profit polynomial for $x = 50$.

92. Manufacturing costs. Ace manufacturing has determined that the cost of labor for producing $x$ transmissions is $0.3x^2 + 400x + 550$ dollars, while the cost of materials is $0.1x^2 + 50x + 800$ dollars.
a) Write a polynomial that represents the total cost of materials and labor for producing $x$ transmissions.
b) Evaluate the total cost polynomial for $x = 500$.
c) Find the cost of labor for 500 transmissions and the cost of materials for 500 transmissions.

93. Perimeter of a triangle. The shortest side of a triangle is $x$ meters, and the other two sides are $3x - 1$ and $2x + 4$ meters. Write a polynomial that represents the perimeter and then evaluate the perimeter polynomial if $x = 4$ meters.

94. Perimeter of a rectangle. The width of a rectangular playground is $2x - 5$ feet, and the length is $3x + 9$ feet. Write a polynomial that represents the perimeter and then evaluate this perimeter polynomial if $x = 4$ feet.

95. Before and after. Jessica traveled $2x + 50$ miles in the morning and $3x - 10$ miles in the afternoon. Write a polynomial that represents the total distance that she traveled. Find the total distance if $x = 20$.

96. Total distance. Hanson drove his rig at $x$ mph for 3 hours, then increased his speed to $x + 15$ mph and drove for 2 more hours. Write a polynomial that represents the total distance that he traveled. Find the total distance if $x = 45$ mph.

97. Sky divers. Bob and Betty simultaneously jump from two airplanes at different altitudes. Bob’s altitude $t$ seconds after leaving the plane is $-16t^2 + 6600$ feet. Betty’s altitude $t$ seconds after leaving the plane is $-16t^2 + 7400$ feet. Write a polynomial that represents the difference between their altitudes $t$ seconds after leaving the planes. What is the difference between their altitudes $3$ seconds after leaving the planes?
4.2 Multiplication of Polynomials

You learned to multiply some polynomials in Chapter 1. In this section you will learn how to multiply any two polynomials.

Multiplying Monomials with the Product Rule

To multiply two monomials, such as \( x^3 \) and \( x^5 \), recall that

\[ x^3 = x \cdot x \cdot x \quad \text{and} \quad x^5 = x \cdot x \cdot x \cdot x \cdot x. \]
So
\[ x^3 \cdot x^5 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{3 \text{ factors}} (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) = x^8. \]

The exponent of the product of \(x^3\) and \(x^5\) is the sum of the exponents 3 and 5. This example illustrates the **product rule** for multiplying exponential expressions.

**Product Rule**

If \(a\) is any real number and \(m\) and \(n\) are any positive integers, then
\[ a^m \cdot a^n = a^{m+n}. \]

**EXAMPLE 1**

Multiplying monomials

Find the indicated products.

a) \(x^2 \cdot x^4 \cdot x\)  

b) \((-2ab)(-3ab)\)  

c) \(-4x^2 y^2 \cdot 3xy^5\)  

d) \((3a)^2\)

**Solution**

a) \(x^2 \cdot x^4 \cdot x = x^2 \cdot x^4 \cdot x^1\)

\[ = x^7 \quad \text{Product rule} \]

b) \((-2ab)(-3ab) = (-2)(-3) \cdot a \cdot a \cdot b \cdot b\)

\[ = 6a^2b^2 \quad \text{Product rule} \]

c) \((-4x^2y^2)(3xy^5) = (-4)(3)x^2 \cdot x \cdot y^2 \cdot y^5\)

\[ = -12x^3y^7 \quad \text{Product rule} \]

d) \((3a)^2 = 3a \cdot 3a\)

\[ = 9a^2 \]

**CAUTION** Be sure to distinguish between adding and multiplying monomials. You can add like terms to get \(3x^4 + 2x^4 = 5x^4\), but you cannot combine the terms in \(3w^5 + 6w^2\). However, you can multiply any two monomials: \(3x^4 \cdot 2x^2 = 6x^6\) and \(3w^5 \cdot 6w^2 = 18w^7\).

**EXAMPLE 2**

**Multiplying Polynomials**

To multiply a monomial and a polynomial, we use the distributive property.

Multiplying monomials and polynomials

Find each product.

a) \(3x^2(x^3 - 4x)\)  

b) \((y^2 - 3y + 4)(-2y)\)  

c) \(-a(b - c)\)

**Solution**

a) \(3x^2(x^3 - 4x) = 3x^2 \cdot x^3 - 3x^2 \cdot 4x\) \quad \text{Distributive property}

\[ = 3x^5 - 12x^3 \]

b) \((y^2 - 3y + 4)(-2y) = y^2(-2y) - 3y(-2y) + 4(-2y)\) \quad \text{Distributive property}

\[ = -2y^3 + 6y^2 - 8y \]

When doing homework or taking notes, use a pencil with an eraser. Everyone makes mistakes. If you get a problem wrong, don’t start over. Check your work for errors and use the eraser. It is better to find out where you went wrong than to simply get the right answer.

**study tip**

As soon as possible after class, find a quiet place and work on your homework. The longer you wait the harder it is to remember what happened in class.
c) \(-a(b - c) = (-a)b - (-a)c\) Distributive property
   
   \[-ab + ac\]
   
   \[= ac - ab\]

Note in part (c) that either of the last two binomials is the correct answer. The last one is just a little simpler to read.

Just as we use the distributive property to find the product of a monomial and a polynomial, we can use the distributive property to find the product of two binomials and the product of a binomial and a trinomial.

**Example 3**

**Multiplying polynomials**

Use the distributive property to find each product.

**a)** \((x + 2)(x + 5)\)  
**b)** \((x + 3)(x^2 + 2x - 7)\)

**Solution**

**a)** First multiply each term of \(x + 5\) by \(x + 2\):

\[
(x + 2)(x + 5) = (x + 2)x + (x + 2)5 \quad \text{Distributive property}
\]

\[
= x^2 + 2x + 5x + 10 \quad \text{Distributive property}
\]

\[
= x^2 + 7x + 10 \quad \text{Combine like terms.}
\]

**b)** First multiply each term of the trinomial by \(x + 3\):

\[
(x + 3)(x^2 + 2x - 7) = (x + 3)x^2 + (x + 3)2x + (x + 3)(-7) \quad \text{Distributive property}
\]

\[
= x^3 + 3x^2 + 2x^2 + 6x - 7x - 21 \quad \text{Distributive property}
\]

\[
= x^3 + 5x^2 - x - 21 \quad \text{Combine like terms.}
\]

Products of polynomials can also be found by arranging the multiplication vertically like multiplication of whole numbers.

**Example 4**

**Multiplying vertically**

Find each product.

**a)** \((x - 2)(3x + 7)\)  
**b)** \((x + y)(a + 3)\)

**Solution**

**a)**

\[
\begin{array}{c}
3x + 7 \\
\hline
x - 2 \\
\hline
-6x - 14 \quad \text{← 2 times } 3x + 7
\end{array}
\]

\[
\begin{array}{c}
3x^2 + 7x \\
\hline
\hline
3x^2 + x - 14 \quad \text{Add.}
\end{array}
\]

**b)**

\[
\begin{array}{c}
x + y \\
\hline
a + 3 \\
\hline
ax + ay \quad \text{← x times } 3x + 7
\end{array}
\]

\[
\begin{array}{c}
ax + ay + 3x + 3y
\end{array}
\]

These examples illustrate the following rule.
**Multiplication of Polynomials**

To multiply polynomials, multiply each term of one polynomial by every term of the other polynomial, then combine like terms.

**The Opposite of a Polynomial**

Note the result of multiplying the difference \( a - b \) by \(-1\):

\[-1(a - b) = -a + b = b - a\]

Because multiplying by \(-1\) is the same as taking the opposite, we can write

\[-(a - b) = b - a.\]

So \( a - b \) and \( b - a \) are opposites or additive inverses of each other. If \( a \) and \( b \) are replaced by numbers, the values of \( a - b \) and \( b - a \) are additive inverses. For example, \(3 - 7 = -4 \) and \(7 - 3 = 4\).

**CAUTION**  The opposite of \( a + b \) is \(-a - b\), not \(a - b\).

**EXAMPLE 5**  Opposite of a polynomial

Find the opposite of each polynomial.

a) \( x - 2 \)  

b) \( 9 - y^2 \)  

c) \( a + 4 \)  

d) \(-x^2 + 6x - 3\)

**Solution**

a) \(- (x - 2) = 2 - x\)

b) \(- (9 - y^2) = y^2 - 9\)

c) \(- (a + 4) = -a - 4\)

d) \(- (-x^2 + 6x - 3) = x^2 - 6x + 3\)

**Applications**

**EXAMPLE 6**  Multiplying polynomials

A parking lot is 20 yards wide and 30 yards long. If the college increases the length and width by the same amount to handle an increasing number of cars, then what polynomial represents the area of the new lot? What is the new area if the increase is 15 yards?

**Solution**

If \( x \) is the amount of increase, then the new lot will be \( x + 20 \) yards wide and \( x + 30 \) yards long as shown in Fig. 4.1. Multiply the length and width to get the area:

\[
(x + 20)(x + 30) = (x + 20)x + (x + 20)30 \\
= x^2 + 20x + 30x + 600 \\
= x^2 + 50x + 600
\]

The polynomial \( x^2 + 50x + 600 \) represents the area of the new lot. If \( x = 15 \), then

\[
x^2 + 50x + 600 = (15)^2 + 50(15) + 600 = 1575.
\]

If the increase is 15 yards, then the area of the lot will be 1575 square yards.
**Warm-ups**

**True or false? Explain your answer.**

1. \(3x^3 \cdot 5x^4 = 15x^{12}\) for any value of \(x\).
2. \(3x^2 \cdot 2x^7 = 5x^9\) for any value of \(x\).
3. \((3y^3)^2 = 9y^6\) for any value of \(y\).
4. \(-3x(5x - 7x^3) = -15x^3 + 21x^2\) for any value of \(x\).
5. \(2x(x^2 - 3x + 4) = 2x^3 - 6x^2 + 8x\) for any number \(x\).
6. \(-2(3 - x) = 2x - 6\) for any number \(x\).
7. \((a + b)(c + d) = ac + ad + bc + bd\) for any values of \(a, b, c,\) and \(d\).
8. \(-(x - 7) = 7 - x\) for any value of \(x\).
9. \(83 - 37 = -(37 - 83)\)
10. The opposite of \(x + 3\) is \(x - 3\) for any number \(x\).

**Exercises**

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

1. What is the product rule for exponents?
2. Why is the sum of two monomials not necessarily a monomial?
3. What property of the real numbers is used when multiplying a monomial and a polynomial?
4. What property of the real numbers is used when multiplying two binomials?
5. How do we multiply any two polynomials?
6. How do we find the opposite of a polynomial?

**Find each product. See Example 1.**

7. \(3x^2 \cdot 9x^3\)
8. \(5x^7 \cdot 3x^5\)
9. \(2a^3 \cdot 7a^8\)
10. \(3y^{12} \cdot 5y^{15}\)
11. \(-6x^2 \cdot 5x^2\)
12. \(-2x^2 \cdot 8x^5\)
13. \((-9x^{10})(-3x^7)\)
14. \((-2x^3)(-8x^6)\)
15. \(-6st \cdot 9st\)
16. \(-12sq \cdot 3s\)
17. \(3wt \cdot 8w^2t^6\)
18. \(h^8k^3 \cdot 5h\)
19. \((5y)^2\)
20. \((6x)^2\)
21. \((2x^3)^2\)
22. \((3y^3)^2\)

**Find each product. See Example 2.**

23. \(4y^2(y^3 - 2y)\)
24. \(6t^2(t^9 + 3t^3)\)
25. \(-3y(6y - 4)\)
26. \(-9y(y^2 - 1)\)
27. \((y^2 - 5y + 6)(-3y)\)
28. \((x^2 - 5x^2 - 17x^2)\)
29. \(-x(y^2 - x^2)\)
30. \(-ab(a^2 - b^2)\)
31. \((3ab^3 - a^2b^2 - 2ab)5a\)
32. \((3c^2d - a^3 + 1)a^3d^2\)
33. \(-\frac{1}{2}r^2(4r^3v^2 - 6rv - 4v)\)
34. \(-\frac{1}{3}m^2n^3(-6mn^2 + 3mn - 12)\)

**Find each product. See Example 3.**

35. \((x + 1)(x + 2)\)
36. \((x + 6)(x + 3)\)
37. \((x - 3)(x + 5)\)
38. \((y - 2)(y + 4)\)
39. \((t - 4)(t - 9)\)
40. \((w - 3)(w - 5)\)
41. \((x + 1)(x^2 + 2x + 2)\)
42. \((x - 1)(x^2 + x + 1)\)
43. \((3y + 2)(2y^2 - y + 3)\)  
44. \((4y + 3)(y^2 + 3y + 1)\)

45. \((y^2z - 2y^4)(y^2z + 3z^2 - y^4)\)

46. \((m^3 - 4mn^2)(6m^2n^2 - 3m + m^2n^4)\)

Find each product vertically. See Example 4.

47. \(2a - 3\) 
    \(a + 5\)

48. \(2w - 6\) 
    \(w + 5\)

49. \(7x + 30\) 
    \(2x + 5\)

50. \(5x + 7\) 
    \(3x + 6\)

51. \(5x + 2\) 
    \(4x - 3\)

52. \(4x + 3\) 
    \(2x - 6\)

53. \(m - 3n\) 
    \(2a + b\)

54. \(3x + 7\) 
    \(a - 2b\)

55. \(x^2 + 3x - 2\) 
    \(x + 6\)

56. \(-x^2 + 3x - 5\) 
    \(x - 7\)

57. \(2a^3 - 3a^2 + 4\) 
    \(-2a - 3\)

58. \(-3x^2 + 5x - 2\) 
    \(-5x - 6\)

59. \(x - y\) 
    \(x + y\)

60. \(a^2 + b^2\) 
    \(a^2 - b^2\)

61. \(x^2 - xy + y^2\) 
    \(x + y\)

62. \(4w^2 + 2wv + v^2\) 
    \(2w - v\)

Find the opposite of each polynomial. See Example 5.

63. \(3t - u\) 
64. \(-3t - u\)

65. \(3x + y\) 
66. \(x - 3y\)

67. \(-3a^2 + a + 6\)

68. \(3b^2 - b - 6\)

69. \(3y^2 + v - 6\)

70. \(-3t^2 + t - 6\)

Perform the indicated operation.

71. \(-3x(2x - 9)\)

72. \(-1(2 - 3x)\)

73. \(2 - 3x(2x - 9)\)

74. \(6 - 3(4x - 8)\)

75. \((2 - 3x) + (2x - 9)\)

76. \((2 - 3x) - (2x - 9)\)

77. \((6x^2)^3\)

78. \((-3a^3)^2\)

79. \(3ab^3(-2a^2b^3)\)

80. \(-4xst \cdot 8x\)

81. \((5x + 6)(5x + 6)\)

82. \((5x - 6)(5x - 6)\)

83. \((5x - 6)(5x + 6)\)

84. \((2x - 9)(2x + 9)\)

85. \(2x^2(3x^2 - 4x^2)\)

86. \(4a^3(3ab^3 - 2ab^3)\)

87. \((m - 1)(m^2 + m + 1)\)

88. \((a + b)(a^2 - ab + b^2)\)

89. \((3x - 2)(x^2 - x - 9)\)

90. \((5 - 6y)(3y^2 - y - 7)\)

Solve each problem. See Example 6.

91. Office space. The length of a professor’s office is \(x\) feet, and the width is \(x + 4\) feet. Write a polynomial that represents the area. Find the area if \(x = 10\) ft.

92. Swimming space. The length of a rectangular swimming pool is \(2x - 1\) meters, and the width is \(x + 2\) meters. Write a polynomial that represents the area. Find the area if \(x = 5\) meters.

93. Area of a truss. A roof truss is in the shape of a triangle with a height of \(x\) feet and a base of \(2x + 1\) feet. Write a polynomial that represents the area of the triangle. What is the area if \(x = 5\) feet?

94. Volume of a box. The length, width, and height of a box are \(x\), \(2x\), and \(3x - 5\) inches, respectively. Write a polynomial that represents its volume.

95. Number pairs. If two numbers differ by 5, then what polynomial represents their product?

96. Number pairs. If two numbers have a sum of 9, then what polynomial represents their product?

97. Area of a rectangle. The length of a rectangle is \(2.3x + 1.2\) meters, and its width is \(3.5x + 5.1\) meters. What polynomial represents its area?
98. **Patchwork.** A quilt patch cut in the shape of a triangle has a base of 5x inches and a height of 1.732x inches. What polynomial represents its area?

![Figure for Exercise 98](image)

99. **Total revenue.** If a promoter charges \( p \) dollars per ticket for a concert in Tulsa, then she expects to sell \( 40,000 - 1000p \) tickets to the concert. How many tickets will she sell if the tickets are $10 each? Find the total revenue when the tickets are $p$ dollars each? What polynomial represents the total revenue expected for the concert when the tickets are $p$ dollars each?

100. **Manufacturing shirts.** If a manufacturer charges \( p \) dollars each for rugby shirts, then he expects to sell \( 2000 - 100p \) shirts per week. What polynomial represents the total revenue expected for a week? How many shirts will be sold if the manufacturer charges $20 each for the shirts? Find the total revenue when the shirts are sold for $p$ each. Use the bar graph to determine the price that will give the maximum total revenue.

101. **Periodic deposits.** At the beginning of each year for 5 years, an investor invests $10 in a mutual fund with an average annual return of \( r \). If we let \( x = 1 + r \), then at the end of the first year (just before the next investment) the value is \( 10x \) dollars. Because $10 is then added to the \( 10x \) dollars, the amount at the end of the second year is \( (10x + 10)x \) dollars. Find a polynomial that represents the value of the investment at the end of the fifth year. Evaluate this polynomial if \( r = 10\% \).

102. **Increasing deposits.** At the beginning of each year for 5 years, an investor invests in a mutual fund with an average annual return of \( r \). The first year, she invests $10; the second year, she invests $20; the third year, she invests $30; the fourth year, she invests $40; the fifth year, she invests $50. Let \( x = 1 + r \) as in Exercise 101 and write a polynomial in \( x \) that represents the value of the investment at the end of the fifth year. Evaluate this polynomial for \( r = 8\% \).

### Getting More Involved

103. **Discussion.** Name all properties of the real numbers that are used in finding the following products:

- (a) \(-2ab^2 \cdot 5a^2bc\)  
- (b) \((x^2 + 3)(x^2 - 8x - 6)\)

104. **Discussion.** Find the product of 27 and 436 without using a calculator. Then use the distributive property to find the product \((20 + 7)(400 + 30 + 6)\) as you would find the product of a binomial and a trinomial. Explain how the two methods are related.

### 4.3 Multiplication of Binomials

In Section 4.2 you learned to multiply polynomials. In this section you will learn a rule that makes multiplication of binomials simpler.

#### The FOIL Method

We can use the distributive property to find the product of two binomials. For example,

\[(x + 2)(x + 3) = (x + 2)x + (x + 2)3\]  
\[= x^2 + 2x + 3x + 6\]  
\[= x^2 + 5x + 6\]  

**Distributive property**

**Distributive property**

**Combine like terms.**
There are four terms in \( x^2 + 2x + 3x + 6 \). The term \( x^2 \) is the product of the first term of each binomial, \( x \) and \( x \). The term \( 3x \) is the product of the two outer terms, \( 3 \) and \( x \). The term \( 2x \) is the product of the two inner terms, \( 2 \) and \( x \). The term \( 6 \) is the product of the last term of each binomial, \( 2 \) and \( 3 \). We can connect the terms multiplied by lines as follows:

\[
(x + 2)(x + 3) \\
\downarrow \quad \downarrow \\
F = \text{First terms} \\
O = \text{Outer terms} \\
I = \text{Inner terms} \\
L = \text{Last terms}
\]

If you remember the word FOIL, you can get the product of the two binomials much faster than writing out all of the steps above. This method is called the **FOIL method**. The name should make it easier to remember.

### Example 1

**Using the FOIL method**

Find each product.

a) \( (x + 2)(x - 4) \)  

\[
(x + 2)(x - 4) = x^2 - 4x + 2x - 8 = x^2 - 2x - 8 \\
\text{Combine the like terms.}
\]

b) \( (2x + 5)(3x - 4) \)

\[
(2x + 5)(3x - 4) = 6x^2 - 8x + 15x - 20 = 6x^2 + 7x - 20 \\
\text{Combine the like terms.}
\]

c) \( (a - b)(2a - b) \)

\[
(a - b)(2a - b) = 2a^2 - ab - 2ab + b^2 = 2a^2 - 3ab + b^2
\]

d) \( (x + 3)(y + 5) \)

\[
(x + 3)(y + 5) = xy + 5x + 3y + 15 \\
\text{There are no like terms to combine.}
\]

FOIL can be used to multiply any two binomials. The binomials in the next example have higher powers than those of Example 1.

### Example 2

**Using the FOIL method**

Find each product.

a) \( (x^3 - 3)(x^2 + 6) \)  

\[
(x^3 - 3)(x^2 + 6) = x^6 + 6x^3 - 3x^3 - 18 \\
= x^6 + 3x^3 - 18
\]

b) \( (2a^2 + 1)(a^2 + 5) \)

\[
(2a^2 + 1)(a^2 + 5) = 2a^4 + 10a^2 + a^2 + 5 \\
= 2a^4 + 11a^2 + 5
\]

**Multiplying Binomials Quickly**

The outer and inner products in the FOIL method are often like terms, and we can combine them without writing them down. Once you become proficient at using FOIL, you can find the product of two binomials without writing anything except the answer.
**Example 3**

**Using FOIL to find a product quickly**

Find each product. Write down only the answer.

a) \((x + 3)(x + 4)\)  
b) \((2x - 1)(x + 5)\)  
c) \((a - 6)(a + 6)\)

**Solution**

a) \((x + 3)(x + 4) = x^2 + 7x + 12\)  
   Combine like terms: \(3x + 4x = 7x\).

b) \((2x - 1)(x + 5) = 2x^2 + 9x - 5\)  
   Combine like terms: \(10x - x = 9x\).

c) \((a - 6)(a + 6) = a^2 - 36\)  
   Combine like terms: \(6a - 6a = 0\).

**Example 4**

**Area of a garden**

Sheila has a square garden with sides of length \(x\) feet. If she increases the length by 7 feet and decreases the width by 2 feet, then what trinomial represents the area of the new rectangular garden?

**Solution**

The length of the new garden is \(x + 7\) and the width is \(x - 2\) as shown in Fig. 4.2. The area is \((x + 7)(x - 2)\) or \(x^2 + 5x - 14\) square feet.

**Warm-Ups**

**True or false? Answer true only if the equation is true for all values of the variable or variables. Explain your answer.**

1. \((x + 3)(x + 2) = x^2 + 6\)
2. \((x + 2)(y + 1) = xy + x + 2y + 2\)
3. \((3a - 5)(2a + 1) = 6a^2 + 3a - 10a - 5\)
4. \((y + 3)(y - 2) = y^2 + y - 6\)
5. \((x^2 + 2)(x^2 + 3) = x^4 + 5x^2 + 6\)
6. \((3a^2 - 2)(3a^2 + 2) = 9a^4 - 4\)
7. \((t + 3)(t + 5) = t^2 + 8t + 15\)
8. \((y - 9)(y - 2) = y^2 - 11y - 18\)
9. \((x + 4)(x - 7) = x^2 + 4x - 28\)
10. It is not necessary to learn FOIL as long as you can get the answer.

**Exercises**

**Reading and Writing**  After reading this section, write out the answers to these questions. Use complete sentences.

1. What property of the real numbers do we usually use to find the product of two binomials?

2. What does FOIL stand for?

3. What is the purpose of FOIL?

4. What is the maximum number of terms that can be obtained when two binomials are multiplied?

4.3 Multiplication of Binomials
10. \((2y - 5)(y - 2)\)
11. \((2a - 3)(a + 1)\)
12. \((3x - 5)(x + 4)\)
13. \((w - 50)(w - 10)\)
14. \((w - 30)(w - 20)\)
15. \((y - a)(y + 5)\)
16. \((a + 1)(3y - 2)\)
17. \((5 - w)(w + m)\)
18. \((a - h)(b + t)\)
19. \((2m - 3t)(5m + 3t)\)
20. \((2x - 5y)(x + y)\)
21. \((5a + 2b)(9a + 7b)\)
22. \((11x + 3y)(x + 4y)\)

Use FOIL to find each product. See Example 2.

23. \((x^2 - 5)(x^2 + 2)\)
24. \((y^2 + 1)(y^3 - 2)\)
25. \((h^3 + 5)(h^3 + 5)\)
26. \((y^6 + 1)(y^6 - 4)\)
27. \((3b^3 + 2)(b^3 + 4)\)
28. \((5n^4 - 1)(n^2 + 3)\)
29. \((y^2 - 3)(y - 2)\)
30. \((x^2 - 3)(x + 1)\)
31. \((3m^3 - n^2)(2m^3 + 3n^2)\)
32. \((6y^4 - 2z^2)(6y^4 - 3z^2)\)
33. \((3u^2v - 2)(4u^2v + 6)\)
34. \((5y^3w^2 + z)(2y^3w^2 + 3z)\)

Find each product. Try to write only the answer. See Example 3.

35. \((b + 4)(b + 5)\)
36. \((y + 8)(y + 4)\)
37. \((x - 3)(x + 9)\)
38. \((m + 7)(m - 8)\)
39. \((a + 5)(a + 5)\)
40. \((t - 4)(t - 4)\)
41. \((2x - 1)(2x - 1)\)
42. \((3y + 4)(3y + 4)\)
43. \((z - 10)(z + 10)\)
44. \((3h - 5)(3h + 5)\)
45. \((a + b)(a + b)\)
46. \((x - y)(x - y)\)
47. \((a - 1)(a - 2)\)
48. \((b - 8)(b - 1)\)
49. \((2x - 1)(x + 3)\)
50. \((3y + 5)(y - 3)\)
51. \((5t - 2)(t - 1)\)
52. \((2t - 3)(2t - 1)\)
53. \((h - 7)(h - 9)\)
54. \((h - 7w)(h - 7w)\)
55. \((h + 7w)(h + 7w)\)
56. \((h - 7q)(h + 7q)\)
57. \((2h^2 - 1)(2h^2 - 1)\)
58. \((3h^2 + 1)(3h^2 + 1)\)

Perform the indicated operations.

59. \(2a + \frac{1}{2} \left( 4a - \frac{1}{2} \right)\)
60. \(3b + \frac{2}{3} \left( 6b - \frac{1}{3} \right)\)
61. \(\left( \frac{1}{2}x - \frac{1}{3} \right) \left( \frac{1}{4}x + \frac{1}{2} \right)\)
62. \(\frac{2}{3}t - \frac{1}{4} \left( \frac{1}{2}t - \frac{1}{2} \right)\)
63. \(-2x^2(3x - 1)(2x + 5)\)
64. \(4xy^3(2x - y)(3x + y)\)
65. \((x - 1)(x + 1)(x + 3)\)
66. \((a - 3)(a + 4)(a - 5)\)
67. \((3x - 2)(3x + 2)(x + 5)\)
68. \((x - 6)(9x + 4)(9x - 4)\)
69. \((x - 1)(x + 2) - (x + 3)(x - 4)\)
70. \((k - 4)(k + 9) - (k - 3)(k + 7)\)

Solve each problem.

71. Area of a rug. Find a trinomial that represents the area of a rectangular rug whose sides are \(x + 3\) feet and \(2x - 1\) feet.

72. Area of a parallelogram. Find a trinomial that represents the area of a parallelogram whose base is \(3x + 2\) meters and whose height is \(2x + 3\) meters.

73. Area of a sail. The sail of a tall ship is triangular in shape with a base of \(4.57x + 3\) meters and a height of \(2.3x - 1.33\) meters. Find a polynomial that represents the area of the triangle.

74. Area of a square. A square has a side of length \(1.732x + 1.414\) meters. Find a polynomial that represents its area.
GETTING MORE INVOLVED

75. Exploration. Find the area of each of the four regions shown in the figure. What is the total area of the four regions? What does this exercise illustrate?

[Figure for Exercise 75]

4.4 SPECIAL PRODUCTS

In Section 4.3 you learned the FOIL method to make multiplying binomials simpler. In this section you will learn rules for squaring binomials and for finding the product of a sum and a difference. These products are called special products.

The Square of a Binomial

To compute \((a + b)^2\), the square of a binomial, we can write it as \((a + b)(a + b)\) and use FOIL:

\[
(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2
\]

So to square \(a + b\), we square the first term \(a^2\), add twice the product of the two terms \(2ab\), then add the square of the last term \(b^2\). The square of a binomial occurs so frequently that it is helpful to learn this new rule to find it. The rule for squaring a sum is given symbolically as follows.

**The Square of a Sum**

\[(a + b)^2 = a^2 + 2ab + b^2\]

**Example 1**

Using the rule for squaring a sum

Find the square of each sum.

a) \((x + 3)^2\)

b) \((2a + 5)^2\)
Chapter 4 Polynomials and Exponents

**Solution**

a) \((x + 3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 6x + 9\)

b) \((2a + 5)^2 = (2a)^2 + 2(2a)(5) + 5^2 = 4a^2 + 20a + 25\)

**CAUTION** Do not forget the middle term when squaring a sum. The equation \((x + 3)^2 = x^2 + 6x + 9\) is an identity, but \((x + 3)^2 = x^2 + 9\) is not an identity. For example, if \(x = 1\) in \((x + 3)^2 = x^2 + 9\), then we get \(4^2 = 1^2 + 9\), which is false.

When we use FOIL to find \((a - b)^2\), we see that
\[
(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2.
\]

So to square \(a - b\), we square the first term \(a^2\), subtract twice the product of the two terms \((-2ab)\), and add the square of the last term \(b^2\). The rule for squaring a difference is given symbolically as follows.

**The Square of a Difference**

\[(a - b)^2 = a^2 - 2ab + b^2\]

**Example 2**

Using the rule for squaring a difference

Find the square of each difference.

a) \((x - 4)^2\)

b) \((4b - 5y)^2\)

**Solution**

a) \((x - 4)^2 = x^2 - 2(x)(4) + 4^2 = x^2 - 8x + 16\)

b) \((4b - 5y)^2 = (4b)^2 - 2(4b)(5y) + (5y)^2 = 16b^2 - 40by + 25y^2\)

**Product of a Sum and a Difference**

If we multiply the sum \(a + b\) and the difference \(a - b\) by using FOIL, we get
\[
(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2.
\]

The inner and outer products have a sum of 0. So the product of a sum and a difference of the same two terms is equal to the difference of two squares.
**Example 3**

Product of a sum and a difference

Find each product.

- **a)** $(x + 2)(x - 2)$
- **b)** $(b + 7)(b - 7)$
- **c)** $(3x - 5)(3x + 5)$

**Solution**

- **a)** $(x + 2)(x - 2) = x^2 - 4$
- **b)** $(b + 7)(b - 7) = b^2 - 49$
- **c)** $(3x - 5)(3x + 5) = 9x^2 - 25$

**Higher Powers of Binomials**

To find a power of a binomial that is higher than 2, we can use the rule for squaring a binomial along with the method of multiplying binomials using the distributive property. Finding the second or higher power of a binomial is called **expanding the binomial** because the result has more terms than the original.

**Example 4**

Higher powers of a binomial

Expand each binomial.

- **a)** $(x + 4)^3$
- **b)** $(y - 2)^4$

**Solution**

- **a)** $(x + 4)^3 = (x + 4)^2(x + 4)$
  $= (x^2 + 8x + 16)(x + 4)$
  $= (x^2 + 8x + 16)x + (x^2 + 8x + 16)4$
  $= x^3 + 8x^2 + 16x + 4x^2 + 32x + 64$
  $= x^3 + 12x^2 + 48x + 64$

- **b)** $(y - 2)^4 = (y - 2)^2(y - 2)^2$
  $= (y^2 - 4y + 4)(y^2 - 4y + 4)$
  $= (y^2 - 4y + 4)(y^2) + (y^2 - 4y + 4)(-4y) + (y^2 - 4y + 4)(4)$
  $= y^4 - 4y^3 + 4y^2 - 4y^3 + 16y^2 - 16y + 4y^2 - 16y + 16$
  $= y^4 - 8y^3 + 24y^2 - 32y + 16$

**Example 5**

**Applications to Area**

**Area of a pizza**

A pizza parlor saves money by making all of its round pizzas one inch smaller in radius than advertised. Write a trinomial for the actual area of a pizza with an advertised radius of $r$ inches.

**Solution**

A pizza advertised as $r$ inches has an actual radius of $r - 1$ inches. The actual area is $\pi(r - 1)^2$:

$$\pi(r - 1)^2 = \pi(r^2 - 2r + 1) = \pi r^2 - 2\pi r + \pi.$$  

So $\pi r^2 - 2\pi r + \pi$ is a trinomial representing the actual area.
**WARM-UPS**

**True or false? Explain your answer.**

1. \((2 + 3)^2 = 2^2 + 3^2\)
2. \((x + 3)^2 = x^2 + 6x + 9\) for any value of \(x\).
3. \((3 + 5)^2 = 9 + 25 + 30\)
4. \((2x + 7)^2 = 4x^2 + 28x + 49\) for any value of \(x\).
5. \((y + 8)^2 = y^2 + 64\) for any value of \(y\).
6. The product of a sum and a difference of the same two terms is equal to the difference of two squares.
7. \((40 - 1)(40 + 1) = 1599\)
8. \(49 \cdot 51 = 2499\)
9. \((x - 3)^2 = x^2 - 3x + 9\) for any value of \(x\).
10. The square of a sum is equal to a sum of two squares.

**EXERCISES**

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

1. What are the special products?
2. What is the rule for squaring a sum?
3. Why do we need a new rule to find the square of a sum when we already have FOIL?
4. What happens to the inner and outer products in the product of a sum and a difference?
5. What is the rule for finding the product of a sum and a difference?
6. How can you find higher powers of binomials?

**Square each binomial. See Example 1.**

7. \((x + 1)^2\)
8. \((y + 2)^2\)
9. \((y + 4)^2\)
10. \((z + 3)^2\)
11. \((3x + 8)^2\)
12. \((2m + 7)^2\)
13. \((s + r)^2\)
14. \((x + z)^2\)
15. \((2x + y)^2\)
16. \((3t + v)^2\)
17. \((2t + 3h)^2\)
18. \((3z + 5k)^2\)
19. \((a - 3)^2\)
20. \((w - 4)^2\)
21. \((t - 1)^2\)
22. \((t - 6)^2\)
23. \((3t - 2)^2\)
24. \((5a - 6)^2\)
25. \((s - t)^2\)
26. \((r - w)^2\)
27. \((3a - b)^2\)
28. \((4w - 7)^2\)
29. \((3z - 5y)^2\)
30. \((2z - 3w)^2\)

**Find each product. See Example 3.**

31. \((a - 5)(a + 5)\)
32. \((x - 6)(x + 6)\)
33. \((y - 1)(y + 1)\)
34. \((p + 2)(p - 2)\)
35. \((3x - 8)(3x + 8)\)
36. \((6x + 1)(6x - 1)\)
37. \((r + s)(r - s)\)
38. \((b - y)(b + y)\)
39. \((8y - 3a)(8y + 3a)\)
40. \((4u - 9v)(4u + 9v)\)
41. \((5x^2 - 2)(5x^2 + 2)\)
42. \((3y^2 + 1)(3y^2 - 1)\)

**Expand each binomial. See Example 4.**

43. \((x + 1)^3\)
44. \((y - 1)^3\)
45. \((2a - 3)^3\)
46. \((3w - 1)^3\)
47. \((a - 3)^4\)
48. \((2b + 1)^4\)
49. \((a + b)^4\)
50. \((2a - 3b)^4\)

Find each product.
51. \((a - 20)(a + 20)\)
52. \((1 - x)(1 + x)\)
53. \((x + 8)(x + 7)\)
54. \((x - 9)(x + 5)\)
55. \((4x - 1)(4x + 1)\)
56. \((9y - 1)(9y + 1)\)
57. \((9y - 1)^2\)
58. \((4x - 1)^2\)
59. \((2t - 5)(3t + 4)\)
60. \((2t + 5)(3t - 4)\)
61. \((2t - 5)^2\)
62. \((2t + 5)^2\)
63. \((2t + 5)(2t - 5)\)
64. \((3t - 4)(3t + 4)\)
65. \((x^2 - 1)(x^2 + 1)\)
66. \((y^3 - 1)(y^3 + 1)\)
67. \((2y^3 - 9)^2\)
68. \((3z^4 - 8)^2\)
69. \((2x^3 + 3y^3)^2\)
70. \((4y^5 + 2w^3)^2\)
71. \(\left(\frac{1}{2x^2} + \frac{1}{3}\right)^2\)
72. \(\left(\frac{2}{3y} - \frac{1}{2}\right)^2\)
73. \((0.2x - 0.1)^2\)
74. \((0.1y + 0.5)^2\)
75. \((a + b)^3\)
76. \((2a - 3b)^3\)
77. \((1.5x + 3.8)^2\)
78. \((3.45a - 2.3)^2\)
79. \((3.5t - 2.5)(3.5t + 2.5)\)
80. \((4.5h + 5.7)(4.5h - 5.7)\)

In Exercises 81–90, solve each problem.

81. **Shrinking garden.** Rose’s garden is a square with sides of length \(x\) feet. Next spring she plans to make it rectangular by lengthening one side 5 feet and shortening the other side by 5 feet. What polynomial represents the new area? By how much will the area of the new garden differ from that of the old garden?

82. **Square lot.** Sam lives on a lot that he thought was a square, 157 feet by 157 feet. When he had it surveyed, he discovered that one side was actually 2 feet longer than he thought and the other was actually 2 feet shorter than he thought. How much less area does he have than he thought he had?

83. **Area of a circle.** Find a polynomial that represents the area of a circle whose radius is \(b + 1\) meters. Use the value 3.14 for \(\pi\).

84. **Comparing dart boards.** A toy store sells two sizes of circular dartboards. The larger of the two has a radius that is 3 inches greater than that of the other. The radius of the smaller dartboard is \(t\) inches. Find a polynomial that represents the difference in area between the two dartboards.

85. **Poiseuille’s law.** According to the nineteenth-century physician Poiseuille, the velocity (in centimeters per second) of blood \(r\) centimeters from the center of an artery of radius \(R\) centimeters is given by

\[ v = k(R - r)(R + r), \]

where \(k\) is a constant. Rewrite the formula using a special product rule.

86. **Going in circles.** A promoter is planning a circular race track with an inside radius of \(r\) feet and a width of \(w\) feet.
89. **Investing in treasury bills.** An investment advisor uses the polynomial $P(1 + r)^{10}$ to predict the value in 10 years of a client’s investment of $P$ dollars with an average annual return $r$. The accompanying graph shows historic average annual returns for the last 20 years for various asset classes (T. Rowe Price, www.troweprice.com). Use the historical average return to predict the value in 10 years of an investment of $10,000 in U.S. treasury bills?

90. **Comparing investments.** How much more would the investment in Exercise 89 be worth in 10 years if the client invests in large company stocks rather than U.S. treasury bills?

---

**FIGURE FOR EXERCISES 89 AND 90**

For Exercise 86

![Figure for Exercise 86](image)

The cost in dollars for paving the track is given by the formula

$$C = 1.2\pi[(r + w)^2 - r^2].$$

Use a special product rule to simplify this formula. What is the cost of paving the track if the inside radius is 1000 feet and the width of the track is 40 feet?

87. **Compounded annually.** $P$ dollars is invested at annual interest rate $r$ for 2 years. If the interest is compounded annually, then the polynomial $P(1 + r)^2$ represents the value of the investment after 2 years. Rewrite this expression without parentheses. Evaluate the polynomial if $P = 200$ and $r = 10\%$.

88. **Compounded semiannually.** $P$ dollars is invested at annual interest rate $r$ for 1 year. If the interest is compounded semiannually, then the polynomial $P\left(1 + \left(\frac{r}{2}\right)^2\right)$ represents the value of the investment after 1 year. Rewrite this expression without parentheses. Evaluate the polynomial if $P = 200$ and $r = 10\%$.

---

**4.5 DIVISION OF POLYNOMIALS**

You multiplied polynomials in Section 4.2. In this section you will learn to divide polynomials.

**Dividing Monomials Using the Quotient Rule**

In Chapter 1 we used the definition of division to divide signed numbers. Because the definition of division applies to any division, we restate it here.

**Division of Real Numbers**

If $a$, $b$, and $c$ are any numbers with $b \neq 0$, then

$$a \div b = c \quad \text{provided that} \quad c \cdot b = a.$$
If \( a \div b = c \), we call \( a \) the **dividend**, \( b \) the **divisor**, and \( c \) (or \( a \div b \)) the **quotient**.

You can find the quotient of two monomials by writing the quotient as a fraction and then reducing the fraction. For example,

\[
x^5 \div x^2 = \frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x}{x} = x^3.
\]

You can be sure that \( x^3 \) is correct by checking that \( x^3 \cdot x^2 = x^5 \). You can also divide \( x^2 \) by \( x^5 \), but the result is not a monomial:

\[
x^2 \div x^5 = \frac{x^2}{x^5} = \frac{1 \cdot x \cdot x}{x \cdot x \cdot x \cdot x} = \frac{1}{x^3}.
\]

Note that the exponent 3 can be obtained in either case by subtracting 5 and 2. These examples illustrate the quotient rule for exponents.

### Quotient Rule

Suppose \( a \neq 0 \), and \( m \) and \( n \) are positive integers.

If \( m \geq n \), then

\[
\frac{a^m}{a^n} = a^{m-n}.
\]

If \( n > m \), then

\[
\frac{a^m}{a^n} = \frac{1}{a^{n-m}}.
\]

Note that if you use the quotient rule to subtract the exponents in \( x^4 \div x^4 \), you get the expression \( x^{4-4} \), or \( x^0 \), which has not been defined yet. Because we must have \( x^4 \div x^4 = 1 \) if \( x \neq 0 \), we define the zero power of a nonzero real number to be 1. We do not define the expression \( 0^0 \).

### Zero Exponent

For any nonzero real number \( a \),

\[
a^0 = 1.
\]

### Example 1

Using the definition of zero exponent

Simplify each expression. Assume that all variables are nonzero real numbers.

**a)** \( 5^0 \) \hspace{1cm} **b)** \( (3xy)^0 \) \hspace{1cm} **c)** \( a^0 + b^0 \)

**Solution**

**a)** \( 5^0 = 1 \) \hspace{1cm} **b)** \( (3xy)^0 = 1 \) \hspace{1cm} **c)** \( a^0 + b^0 = 1 + 1 = 2 \)

With the definition of zero exponent the quotient rule is valid for all positive integers as stated.

### Example 2

Using the quotient rule in dividing monomials

Find each quotient.

**a)** \( \frac{y^9}{y^5} \) \hspace{1cm} **b)** \( \frac{12b^2}{3b^7} \) \hspace{1cm} **c)** \( -6x^3 \div (2x^9) \) \hspace{1cm} **d)** \( \frac{x^8y^2}{x^3y^2} \)
Solution

a) \( \frac{y^9}{y^5} = y^{9-5} = y^4 \)

Use the definition of division to check that \( y^4 \cdot y^5 = y^9 \).

b) \( \frac{12b^2}{3b^7} = \frac{12}{3} \cdot \frac{b^2}{b^7} = 4 \cdot \frac{1}{b^{7-2}} = \frac{4}{b^5} \)

Use the definition of division to check that \( \frac{4}{b^5} \cdot 3b^7 = \frac{12b^7}{b^5} = 12b^2 \).

c) \(-6x^3 \div (2x^9) = \frac{-6x^3}{2x^9} = \frac{-3}{x^6} \)

Use the definition of division to check that \( \frac{-3}{x^6} \cdot 2x^9 = \frac{-6x^9}{x^6} = -6x^3 \).

d) \( \frac{x^8y^2}{x^2y^3} = \frac{x^8}{x^2} \cdot \frac{y^2}{y^3} = x^{6} \cdot y^{0} = x^6 \)

Use the definition of division to check that \( x^6 \cdot x^2y^2 = x^8y^2 \).

We showed more steps in Example 2 than are necessary. For division problems like these you should try to write down only the quotient.

Dividing a Polynomial by a Monomial

We divided some simple polynomials by monomials in Chapter 1. For example,

\[
\frac{6x + 8}{2} = \frac{1}{2} (6x + 8) = 3x + 4.
\]

We use the distributive property to take one-half of 6x and one-half of 8 to get 3x + 4. So both 6x and 8 are divided by 2. To divide any polynomial by a monomial, we divide each term of the polynomial by the monomial.

EXAMPLE 3

Dividing a polynomial by a monomial

Find the quotient for \((-8x^6 + 12x^4 - 4x^2) \div (4x^2)\).

Solution

\[
\frac{-8x^6 + 12x^4 - 4x^2}{4x^2} = \frac{-8x^6}{4x^2} + \frac{12x^4}{4x^2} - \frac{4x^2}{4x^2} = -2x^4 + 3x^2 - 1
\]

The quotient is \(-2x^4 + 3x^2 - 1\). We can check by multiplying.

\[
4x^2(-2x^4 + 3x^2 - 1) = -8x^6 + 12x^4 - 4x^2.
\]

Because division by zero is undefined, we will always assume that the divisor is nonzero in any quotient involving variables. For example, the division in Example 3 is valid only if \(4x^2 \neq 0\), or \(x \neq 0\).
Dividing a Polynomial by a Binomial

Division of whole numbers is often done with a procedure called long division. For example, 253 is divided by 7 as follows:

\[
\begin{array}{l}
\text{Divisor} \to \quad 36 \\
7)253 \\
\text{Quotient} \\
21 \\
43 \\
42 \\
1 \\
\text{Dividend} \\
\text{Remainder}
\end{array}
\]

Note that \(36 \cdot 7 + 1 = 253\). It is always true that

\[(\text{quotient})(\text{divisor}) + (\text{remainder}) = \text{dividend}.
\]

To divide a polynomial by a binomial, we perform the division like long division of whole numbers. For example, to divide \(x^2 - 3x - 10\) by \(x + 2\), we get the first term of the quotient by dividing the first term of \(x + 2\) into the first term of \(x^2 - 3x - 10\). So divide \(x^2\) by \(x\) to get \(x\), then multiply and subtract as follows:

1. Divide: \(x\)
2. Multiply: \(x^2 + 2x\)
3. Subtract: \(-5x - 10\)

Now bring down \(-10\) and continue the process. We get the second term of the quotient (below) by dividing the first term of \(x + 2\) into the first term of \(-5x - 10\). So divide \(-5x\) by \(x\) to get \(-5\):

1. Divide: \(-5\)
2. Multiply: \(x^2 + 2x\)
3. Subtract: \(-10 - (-10) = 0\)

So the quotient is \(x - 5\), and the remainder is 0.

In the next example we must rearrange the dividend before dividing.

**Example 4**

**Dividing a polynomial by a binomial**

Divide \(2x^3 - 4 - 7x^2\) by \(2x - 3\), and identify the quotient and the remainder.

**Solution**

Rearrange the dividend as \(2x^3 - 7x^2 - 4\). Because the \(x\)-term in the dividend is missing, we write \(0 \cdot x\) for it:

\[
\begin{array}{l}
2x^3 - 7x^2 - 4 \\
\frac{x^2 - 2x - 3}{2x - 3} \\
\frac{2x^3 - 3x^2}{2x^3 - 7x^2 + 0 \cdot x - 4} \\
\frac{-4x^2 + 0 \cdot x}{-4x^2 + 6x} \\
\frac{-6x - 4}{-6x + 9} \\
-13
\end{array}
\]

So the quotient is \(x^2 - 2x - 3\) and the remainder is 0.
The quotient is \( x^2 - 2x - 3 \), and the remainder is \(-13\). Note that the degree of the remainder is 0 and the degree of the divisor is 1. To check, we must verify that
\[
(2x - 3)(x^2 - 2x - 3) - 13 = 2x^3 - 7x^2 - 4.
\]

**CAUTION** To avoid errors, always write the terms of the divisor and the dividend in descending order of the exponents and insert a zero for any term that is missing.

If we divide both sides of the equation
\[
\text{dividend} = (\text{quotient})(\text{divisor}) + (\text{remainder})
\]
by the divisor, we get the equation
\[
\frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}.
\]

This fact is used in expressing improper fractions as mixed numbers. For example, if 19 is divided by 5, the quotient is 3 and the remainder is 4. So
\[
\frac{19}{5} = 3 + \frac{4}{5} = 3\frac{4}{5}.
\]

We can also use this form to rewrite algebraic fractions.

**EXAMPLE 5**

**Rewriting algebraic fractions**

Express \( -\frac{3x}{x - 2} \) in the form
\[
\text{quotient} + \frac{\text{remainder}}{\text{divisor}}.
\]

**Solution**

Use long division to get the quotient and remainder:

\[
\begin{array}{c|cc}
& -3 & \\
\hline
x - 2 & -3x & + 0 \\
& -3x & + 6 \\
\hline
& -6
\end{array}
\]

Because the quotient is \(-3\) and the remainder is \(-6\), we can write
\[
-\frac{3x}{x - 2} = -3 + \frac{-6}{x - 2}.
\]

To check, we must verify that \(-3(x - 2) - 6 = -3x\).

**CAUTION** When dividing polynomials by long division, we do not stop until the remainder is 0 or the degree of the remainder is smaller than the degree of the divisor. For example, we stop dividing in Example 5 because the degree of the remainder \(-6\) is 0 and the degree of the divisor \(x - 2\) is 1.

**WARM-UPS**

**True or false? Explain your answer.**

1. \( y^{10} \div y^2 = y^5 \) for any nonzero value of \( y \).
2. \( \frac{7x + 2}{7} = x + 2 \) for any value of \( x \).
3. \( \frac{7x^2}{7} = x^2 \) for any value of \( x \).
4. If \( 3x^2 + 6 \) is divided by 3, the quotient is \( x^2 + 6 \).
5. If $4y^2 - 6y$ is divided by $2y$, the quotient is $2y - 3$.
6. The quotient times the remainder plus the dividend equals the divisor.

7. $(x + 2)(x + 1) + 3 = x^2 + 3x + 5$ for any value of $x$.
8. If $x^2 + 3x + 5$ is divided by $x + 2$, then the quotient is $x + 1$.
9. If $x^2 + 3x + 5$ is divided by $x + 2$, the remainder is $3$.
10. If the remainder is zero, then $(\text{divisor})(\text{quotient}) = \text{dividend}$.

http://www.sosmath.com/algebra/factor/fac01/fac01.html

4.5 Exercises

4.5 Division of Polynomials

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.
1. What rule is important for dividing monomials?
2. What is the meaning of a zero exponent?
3. How many terms should you get when dividing a polynomial by a monomial?
4. How should the terms of the polynomials be written when dividing with long division?
5. How do you know when to stop the process in long division of polynomials?
6. How do you handle missing terms in the dividend polynomial when doing long division?

Simplify each expression. See Example 1.
7. $9^0$  
8. $m^0$
9. $(-2x^3)^0$
10. $(5a^3b)^0$
11. $2 \cdot 5^0 - 3^0$
12. $-4^0 - 8^0$
13. $(2x - y)^0$
14. $(a^2 + b^3)^0$

Find each quotient. Try to write only the answer. See Example 2.
15. $\frac{x^3}{x^2}$  
16. $\frac{y^6}{y^3}$
17. $\frac{6a^7}{2a^{12}}$
18. $\frac{30b^2}{3b^6}$
19. $-12x^5 + (3x^9)$
20. $-6y^2 + (-3y^{10})$
21. $-6y^2 + (6y)$
22. $-3a^2b + (3ab)$
23. $\frac{-6x^3y^2}{2x^3y^2}$
24. $\frac{-4h^2k^4}{-2hk^3}$
25. $\frac{-9x^2y^2}{3x^5y^2}$
26. $\frac{-12z^4y^2}{-2e^{10}y^2}$

Find the quotients. See Example 3.
27. $\frac{3x - 6}{3}$
28. $\frac{5y - 10}{-5}$
29. $\frac{x^5 + 3x^4 - x^3}{x^2}$
30. $\frac{6y^6 - 9y^4 + 12y^2}{3y^2}$
31. $\frac{-8x^2y^2 + 4x^3y - 2xy^2}{-2xy}$
32. $\frac{-9ab^2 - 2a^3b^3}{-3ab^2}$
33. $(x^3y^3 - 3x^4y^2) + (x^4y)$
34. $(4h^5k - 6h^3k^2) + (-2h^2k)$

Find the quotient and remainder for each division. Check by using the fact that dividend = (divisor)(quotient) + remainder. See Example 4.
35. $(x^2 + 5x + 13) \div (x + 3)$
36. $(x^3 + 3x + 6) \div (x + 3)$
37. $(2x) \div (x + 5)$
38. $(5x) \div (x - 1)$
39. $(a^3 + 4a - 3) \div (a - 2)$
40. $(w^3 + 2w^2 - 3) \div (w - 2)$
41. $(x^2 - 3x) \div (x + 1)$
42. $(3x^2) \div (x + 1)$
43. $(h^3 - 27) \div (h - 3)$
44. $(w^3 + 1) \div (w + 1)$
45. $(6x^2 - 13x + 7) \div (x^2 - 2)$
46. $(4b^2 + 25b - 3) \div (4b + 1)$
47. $(x^3 - x^2 + x - 2) \div (x - 1)$
48. $(a^3 - 3a^2 + 4a - 4) \div (a - 2)$
Write each expression in the form
$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$ See Example 5.

49. \(\frac{3x}{x - 5}\) 
50. \(\frac{2x}{x - 1}\)

51. \(\frac{-x}{x + 3}\) 
52. \(\frac{-3x}{x + 1}\)

53. \(\frac{x - 1}{x}\) 
54. \(\frac{a - 5}{a}\)

55. \(\frac{3x + 1}{x}\) 
56. \(\frac{2y + 1}{y}\)

57. \(\frac{x^2}{x + 1}\) 
58. \(\frac{x^2}{x - 1}\)

59. \(\frac{x^2 + 4}{x + 2}\) 
60. \(\frac{x^2 + 1}{x - 1}\)

61. \(\frac{x^3}{x - 2}\) 
62. \(\frac{x^3 - 1}{x + 1}\)

63. \(\frac{x^3 + 3}{x}\) 
64. \(\frac{2x^2 + 4}{2x}\)

Find each quotient.

65. \(-6a^2b + (2a^2b)\) 
66. \(-14x^3 + (-7x^2)\)

67. \(-8w^3t^6 + (-2w^3t^3)\)

68. \(-9y^7z^4 + (3y^3z^4)\)

69. \((3a - 12) + (-3)\)

70. \((-6x + 3x^2) + (-3z)\)

71. \((3x^2 - 9x) ÷ (3x)\)

72. \((5x^3 + 15x^2 - 25x) ÷ (5x)\)

73. \((12x^4 - 4x^3 + 6x^2) ÷ (-2x^2)\)

74. \((-9x^3 + 3x^2 - 15x) ÷ (-3x)\)

75. \((t^2 - 5t - 36) ÷ (t - 9)\)

76. \((b^2 + 2b - 35) ÷ (b - 5)\)

77. \((6w^2 - 7w - 5) ÷ (3w - 5)\)

78. \((4z^2 + 23z - 6) ÷ (4z - 1)\)

79. \((8y^3 + 27) ÷ (2x + 3)\)

80. \((8y^3 - 1) ÷ (2y - 1)\)

81. \((t^3 - 3t^2 + 5t - 6) ÷ (t - 2)\)

82. \((2u^3 - 13u^2 - 8u + 7) ÷ (u - 7)\)

83. \((-6v^2 - 4 + 9v + v^3) ÷ (v - 4)\)

84. \((14y + 8y^2 + y^3 + 12) ÷ (6 + y)\)

Solve each problem.

85. \textit{Area of a rectangle.} The area of a rectangular billboard is \(x^2 + x - 30\) square meters. If the length is \(x + 6\) meters, find a binomial that represents the width.

![FIGURE FOR EXERCISE 85](image)

86. \textit{Perimeter of a rectangle.} The perimeter of a rectangular backyard is \(6x + 6\) yards. If the width is \(x\) yards, find a binomial that represents the length.

![FIGURE FOR EXERCISE 86](image)

87. \textit{Exploration.} Divide \(x^3 - 1\) by \(x - 1\), \(x^4 - 1\) by \(x - 1\), and \(x^5 - 1\) by \(x - 1\). What is the quotient when \(x^3 - 1\) is divided by \(x - 1\)?

88. \textit{Exploration.} Divide \(a^3 - b^3\) by \(a - b\) and \(a^4 - b^4\) by \(a - b\). What is the quotient when \(a^3 - b^3\) is divided by \(a - b\)?

89. \textit{Discussion.} Are the expressions \(\frac{10x}{5x}\), \(10x ÷ 5x\), and \((10x) ÷ (5x)\) equivalent? Before you answer, review the order of operations in Section 1.5 and evaluate each expression for \(x = 3\).
4.6 Positive Integral Exponents

The product rule for positive integral exponents was presented in Section 4.2, and the quotient rule was presented in Section 4.5. In this section we review those rules and then further investigate the properties of exponents.

**The Product and Quotient Rules**

The rules that we have already discussed are summarized below.

The following rules hold for nonnegative integers \( m \) and \( n \) and \( a \neq 0 \).

- **Product rule**
  \[ a^m \cdot a^n = a^{m+n} \]
- **Quotient rule**
  \[ \frac{a^m}{a^n} = a^{m-n} \quad \text{if} \quad m \geq n \]
- **Power of a Product**
  \[ a^m \cdot a^n = a^{m+n} \]
- **Power of a Quotient**
  \[ \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad \text{if} \quad n \geq m \]
- **Zero exponent**
  \[ a^0 = 1 \]

**CAUTION** The product and quotient rules apply only if the bases of the expressions are identical. For example, \( 3^2 \cdot 3^4 = 3^6 \), but the product rule cannot be applied to \( 5^2 \cdot 3^4 \). Note also that the bases are not multiplied: \( 3^2 \cdot 3^4 = 96 \).

Note that in the quotient rule the exponents are always subtracted, as in
\[ \frac{x^7}{x^3} = x^4 \quad \text{and} \quad \frac{y^5}{y^3} = \frac{1}{y^2} \]

If the larger exponent is in the denominator, then the result is placed in the denominator.

**Example 1**

Using the product and quotient rules

Use the rules of exponents to simplify each expression. Assume that all variables represent nonzero real numbers.

\begin{align*}
\text{a)} & \quad 2^3 \cdot 2^2 \\
\text{b)} & \quad (3x)^0(5x^3)(4x) \\
\text{c)} & \quad \frac{8x^2}{-2x^5} \\
\text{d)} & \quad (3a^2b)^0(6a^3b^2)
\end{align*}

**Solution**

\begin{align*}
\text{a)} & \quad \text{Because the bases are both 2, we can use the product rule:} \\
& \quad 2^3 \cdot 2^2 = 2^5 \\
& \quad = 32 & \text{Product rule} \\
& \quad \text{Simplify.} \\
\text{b)} & \quad (3x)^0(5x^3)(4x) = 1 \cdot 5x^2 \cdot 4x \\
& \quad = 20x^3 & \text{Definition of zero exponent} \\
& \quad \text{Product rule} \\
\text{c)} & \quad \frac{8x^2}{-2x^5} = \frac{-4}{x^3} & \text{Quotient rule}
\end{align*}
d) First use the product rule to simplify the numerator and denominator:

\[
\frac{(3a^2b)b^9}{(6a^3)ab^2} = \frac{3a^2b^{10}}{6a^4b^2} \quad \text{Product rule}
\]

\[
= \frac{b^8}{2a^6} \quad \text{Quotient rule}
\]

### Raising an Exponential Expression to a Power

When we raise an exponential expression to a power, we can use the product rule to find the result, as shown in the following example:

\[
(w^4)^3 = w^4 \cdot w^4 \cdot w^4 \quad \text{Three factors of } w^4 \text{ because of the exponent } 3
\]

\[
= w^{12} \quad \text{Product rule}
\]

By the product rule we add the three 4’s to get 12, but 12 is also the product of 4 and 3. This example illustrates the **power rule** for exponents.

### Power Rule

If \(m\) and \(n\) are nonnegative integers and \(a \neq 0\), then

\[
(a^m)^n = a^{mn}.
\]

In the next example we use the new rule along with the other rules.

### Example 2

**Using the power rule**

Use the rules of exponents to simplify each expression. Assume that all variables represent nonzero real numbers.

**a)** \(3x^2(x^3)^5\)

**b)** \(\frac{(2^3)^4 \cdot 2^7}{2^5 \cdot 2^9}\)

**c)** \(\frac{3(x^5)^4}{15x^{22}}\)

**Solution**

**a)** \(3x^2(x^3)^5 = 3x^2x^{15} = 3x^{17} \quad \text{Power rule}
\]

\(= 3x^{17} \quad \text{Product rule}
\]

**b)** \(\frac{(2^3)^4 \cdot 2^7}{2^5 \cdot 2^9} = \frac{2^{12} \cdot 2^7}{2^{14}} = \frac{2^{19}}{2^{14}} = 2^5 = 32 \quad \text{Power rule and product rule}
\]

\(= \frac{2^{19}}{2^{14}} \quad \text{Quotient rule}
\]

\(= 2^5 \quad \text{Evaluate } 2^5.
\]

**c)** \(\frac{3(x^5)^4}{15x^{22}} = \frac{3x^{20}}{15x^{22}} = \frac{1}{5x^2} \quad \text{Power rule and product rule}
\]

\(= \frac{3x^{20}}{15x^{22}} = \frac{1}{5x^2} \quad \text{Quotient rule}
\]

\(= 32 \quad \text{Evaluate } 2^5.
\]

### Power of a Product

Consider an example of raising a monomial to a power. We will use known rules to rewrite the expression.

\[
(2x)^3 = 2x \cdot 2x \cdot 2x \quad \text{Definition of exponent } 3
\]

\[
= 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \quad \text{Commutative and associative properties}
\]

\[
= 2^3x^3 \quad \text{Definition of exponents}
\]
Note that the power 3 is applied to each factor of the product. This example illustrates the **power of a product rule**.

### Power of a Product Rule

If $a$ and $b$ are real numbers and $n$ is a positive integer, then

$$(ab)^n = a^n b^n.$$ 

**Example 3**

Using the **power of a product rule**

Simplify. Assume that the variables are nonzero.

a) $(xy)^5$  

b) $(-3m)^3$  

c) $(2x^3 y^2 z^7)^3$

**Solution**

a) $(xy)^5 = x^5(y^3)^5$  

Power of a product rule  

$= x^5y^{15}$  

Power rule

b) $(-3m)^3 = (-3)^3 m^3$  

Power of a product rule  

$= -27m^3$  

$(-3)(-3)(-3) = -27$

c) $(2x^3 y^2 z^7)^3 = 2^3(x^3)^3(y^2)^3(z^7)^3 = 8x^9y^6z^{21}$

### Power of a Quotient

Raising a quotient to a power is similar to raising a product to a power:

$$
\left( \frac{x}{5} \right)^3 = \frac{x^3}{5^3} \quad \text{Definition of exponent 3}
$$

$$
= \frac{x \cdot x \cdot x}{5 \cdot 5 \cdot 5} \quad \text{Definition of multiplication of fractions}
$$

$$
= \frac{x^3}{5^3} \quad \text{Definition of exponents}
$$

The power is applied to both the numerator and denominator. This example illustrates the **power of a quotient rule**.

### Example 4

Using the **power of a quotient rule**

Simplify. Assume that the variables are nonzero.

a) $\left( \frac{2}{5x^3} \right)^2$  

b) $\left( \frac{3x^4}{2y^3} \right)^3$  

c) $\left( \frac{-12a^3b}{4a^2b^7} \right)^3$

**Solution**

a) $\left( \frac{2}{5x^3} \right)^2 = \frac{2^2}{(5x^3)^2}$  

Power of a quotient rule  

$= \frac{4}{25x^6}$  

$(5x)^2 = 5^2(x^3)^2 = 25x^6$
b) \[
\left(\frac{3x^4}{2y^3}\right)^3 = \frac{3^3x^{12}}{2^3y^9} = \frac{27x^{12}}{8y^9}
\]
Power of a quotient and power of a product rules
Simplify.

\[
\left(\frac{-12a^6b^2}{4a^3b^5}\right)^3 = \left(\frac{-3a^3}{b^2}\right)^3
\]
Use the quotient rule first.
\[
= \frac{-27a^9}{b^6}
\]
Power of a quotient rule

**Summary of Rules**

The rules for exponents are summarized in the following box.

**Rules for Nonnegative Integral Exponents**

The following rules hold for nonzero real numbers \(a\) and \(b\) and nonnegative integers \(m\) and \(n\).

1. \(a^0 = 1\)  
   Definition of zero exponent
2. \(a^m \cdot a^n = a^{m+n}\)  
   Product rule
3. \(\frac{a^m}{a^n} = a^{m-n}\) for \(m \geq n\),
   \(\frac{a^m}{a^n} = \frac{1}{a^{n-m}}\) for \(n > m\)  
   Quotient rule
4. \((a^m)^n = a^{mn}\)  
   Power rule
5. \((ab)^n = a^n \cdot b^n\)  
   Power of a product rule
6. \(\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\)  
   Power of a quotient rule

**WARM-UPS**

True or false? Assume that all variables represent nonzero real numbers. A statement involving variables is to be marked true only if it is an identity. Explain your answer.

1. \(-3^0 = 1\)
2. \(2^5 \cdot 2^8 = 4^{13}\)
3. \(2^3 \cdot 3^1 = 6^5\)
4. \((2x)^4 = 2x^4\)
5. \((q^3)^5 = q^8\)
6. \((-3x^2)^3 = 27x^6\)
7. \((ab^3)^4 = a^4b^{12}\)
8. \(\frac{a^{12}}{a^3} = a^3\)
9. \(6w^4 \div 3w^9 = 2w^5\)
10. \(\frac{(2y^3)^2}{9} = \frac{4y^6}{81}\)
4.6 Exercises

Reading and Writing  After reading this section, write out the answers to these questions. Use complete sentences.

1. What is the product rule for exponents?

2. What is the quotient rule for exponents?

3. Why must the bases be the same in these rules?

4. What is the power rule for exponents?

5. What is the power of a product rule?

6. What is the power of a quotient rule?

For all exercises in this section, assume that the variables represent nonzero real numbers.

Simplify the exponential expressions. See Example 1.

7. $2^3 \cdot 2^5$
8. $x^6 \cdot x^7$
9. $(-3a^2)(-2a^3)$
10. $(3y^3)(-6r^2)$
11. $a^5b^4 \cdot ab^5(ab)^0$
12. $x^2y \cdot x^3y^6(x + y)^9$
13. $-\frac{2a^3}{4d}$
14. $\frac{-3t^6}{6t^{18}}$
15. $\frac{2a^3b \cdot 3a^3b^3}{15a^6b^8}$
16. $\frac{3xy^8 \cdot 5xy^9}{20x^2y^{14}}$
17. $2^3 \cdot 5^2$
18. $2^2 \cdot 10^3$

Simplify. See Example 2.

19. $(x^2)^3$
20. $(y^2)^4$
21. $2x^3 \cdot (x^2)^5$
22. $(y^2)^6 \cdot 3y^5$
23. $\frac{(r^2)^3}{(r^3)^4}$
24. $\frac{(x^2y^2)^3}{(xy^3)^2}$
25. $\frac{3x(x^2y)^2}{6x^3(x^2)^4}$

Simplify. See Example 3.

26. $\frac{5y^2(y^5)^2}{10y^5(y^2)^6}$
27. $(xy^2)^3$
28. $(w^2y^2)^6$
29. $(-2r^2)^3$
30. $(-3x^4)^3$
31. $(-2x^3y^5)^3$
32. $(-3y^2z^2)^3$
33. $\frac{(a^2b^2c^2)^3}{a^2b^4c}$
34. $\frac{(2ab^2c)^3}{(2a^2bc)^3}$
35. $\left(\frac{x^4}{4}\right)^3$
36. $\left(\frac{y^2}{2}\right)^3$
37. $\left(\frac{-2a^2}{b^3}\right)^4$
38. $\left(\frac{-9x^3}{t}\right)^2$
39. $\left(\frac{2x^3y}{-4y^7}\right)^3$
40. $\left(\frac{3y^8}{2y^7}\right)^4$
41. $\left(\frac{-6x^3z^{10}}{3x^5y^4z^3}\right)^2$
42. $\left(\frac{-10rs^0t^1}{2rs^2t^3}\right)^3$

Simplify each expression. Your answer should be an integer or a fraction. Do not use a calculator.

43. $3^2 + 6^2$
44. $(5 - 3)^2$
45. $(3 + 6)^2$
46. $5^2 - 3^2$
47. $2^3 - 3^1$
48. $3^3 + 4^3$
49. $(2 - 3)^3$
50. $(3 + 4)^3$
51. $\left(\frac{2}{3}\right)^3$
52. $\left(\frac{3^3}{4}\right)$
53. $5^2 \cdot 2^3$
54. $10^3 \cdot 3^3$
55. $2^3 \cdot 2^4$
56. $10^2 \cdot 10^4$
57. $\left(\frac{2^3}{2^2}\right)^2$
58. $\left(\frac{3^2}{3^2}\right)$

Simplify each expression.

59. $3x^4 \cdot 5x^7$
60. $-2y^3(3y)$
61. $(-5x^4)^3$
62. $(4x^3)^2$
63. $-3y^5z^{12} \cdot 9y^2z^3$
64. $2a^5b^5 \cdot 2a^9b^2$
65. $-9u^6v^9$
66. $\frac{-20a^4b^{13}}{5a^4b^{13}}$
67. $(-x^2t)(-2x^2t)^4$
68. $(ab)^2(-3ab)^4$
69. $\frac{2x^3}{x^2}$
70. $\left(\frac{3y^3}{x^3}\right)^2$
71. $\left(\frac{-8a^3b^4}{4c^5}\right)^5$
72. $\left(-\frac{10ac}{5a^2b^4}\right)^5$
73. $\left(\frac{-8x^3y}{-16x^2y^5}\right)^5$
74. $\left(-\frac{5x^3y^3z}{-5x^2y^{5}}\right)^5$

Solve each problem.

75. Long-term investing. Sheila invested $P$ dollars at annual rate $r$ for 10 years. At the end of 10 years her investment was worth $P(1 + r)^{10}$ dollars. She then reinvested this money for another 5 years at annual rate $r$. At the end of the second time period her investment was worth...
$P(1 + r)^{10}(1 + r)^5$ dollars. Which law of exponents can be used to simplify the last expression? Simplify it.

76. **CD rollover.** Ronnie invested $P$ dollars in a 2-year CD with an annual rate of return of $r$. After the CD rolled over two times, its value was $P((1 + r)^3)^2$. Which law of exponents can be used to simplify the expression? Simplify it.

---

### GETTING MORE INVOLVED

77. **Writing.** When we square a product, we square each factor in the product. For example, $(3b)^2 = 9b^2$. Explain why we cannot square a sum by simply squaring each term of the sum.

78. **Writing.** Explain why we define $2^0$ to be 1. Explain why $-2^0 \neq 1$.

---

### 4.7 N E G A T I V E E X P O N E N T S A N D S C I E N T I F I C N O T A T I O N

We defined exponential expressions with positive integral exponents in Chapter 1 and learned the rules for positive integral exponents in Section 4.6. In this section you will first study negative exponents and then see how positive and negative integral exponents are used in scientific notation.

#### Negative Integral Exponents

If $x$ is nonzero, the reciprocal of $x$ is written as $\frac{1}{x}$. For example, the reciprocal of $2^3$ is written as $\frac{1}{2^3}$. To write the reciprocal of an exponential expression in a simpler way, we use a negative exponent. So $2^{-3} = \frac{1}{8}$. In general we have the following definition.

**Negative Integral Exponents**

If $a$ is a nonzero real number and $n$ is a positive integer, then

$$a^{-n} = \frac{1}{a^n} \quad (\text{If } n \text{ is positive, } -n \text{ is negative.})$$

#### Example 1

**Simplifying expressions with negative exponents**

Simplify.

**a)** $2^{-5}$

**b)** $(-2)^{-5}$

**c)** $\frac{2^{-3}}{3^{-2}}$

**Solution**

**a)** $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

**b)** $(-2)^{-5} = \frac{1}{(-2)^5} = \frac{1}{-32} = -\frac{1}{32}$

**c)** $\frac{2^{-3}}{3^{-2}} = 2^{-3} \div 3^{-2}$

$$= \frac{1}{2^3} \div \frac{1}{3^2} = \frac{1}{8} \div \frac{1}{9} = \frac{9}{8} = \frac{9}{8}$$
EXAMPLE 2

Using the rules for negative exponents

Note that \(-5^{-2} = \frac{1}{(-5)^2} = \frac{1}{25}\).

To evaluate \(a^{-n}\), you can first find the \(n\)th power of \(a\) and then find the reciprocal. However, the result is the same if you first find the reciprocal of \(a\) and then find the \(n\)th power of the reciprocal. For example,

\[ 3^{-2} = \frac{1}{3^2} = \frac{1}{9} \quad \text{or} \quad \left(\frac{1}{3}\right)^2 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}. \]

*So the power and the reciprocal can be found in either order.* If the exponent is \(-1\), we simply find the reciprocal. For example,

\[ 5^{-1} = \frac{1}{5}, \quad \left(\frac{1}{4}\right)^{-1} = 4, \quad \text{and} \quad \left(-\frac{3}{5}\right)^{-1} = -\frac{5}{3}. \]

Because \(3^{-2} \cdot 3^{2} = 1\), the reciprocal of \(3^{-2}\) is \(3^2\), and we have

\[ \frac{1}{3^{-2}} = 3^2. \]

These examples illustrate the following rules.

**Rules for Negative Exponents**

If \(a\) is a nonzero real number and \(n\) is a positive integer, then

\[ a^{-n} = \left(\frac{1}{a}\right)^n, \quad a^{-1} = \frac{1}{a}, \quad \frac{1}{a^{-n}} = a^n, \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n. \]

E X A M P L E 2

Using the rules for negative exponents

Simplify.

a) \( \left(\frac{3}{4}\right)^{-3} \)

b) \( 10^{-1} + 10^{-1} \)

c) \( \frac{2}{10^{-3}} \)

**Solution**

a) \( \left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^3 = \frac{64}{27} \)

b) \( 10^{-1} + 10^{-1} = \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5} \)

c) \( \frac{2}{10^{-3}} = 2 \cdot \frac{1}{10^{-3}} = 2 \cdot 10^3 = 2 \cdot 1000 = 2000 \)

**Rules for Integral Exponents**

Negative exponents are used to make expressions involving reciprocals simpler looking and easier to write. Negative exponents have the added benefit of working in conjunction with all of the rules of exponents that you learned in Section 4.6. For example, we can use the product rule to get

\[ x^{-2} \cdot x^{-3} = x^{-2+(-3)} = x^{-5} \]

and the quotient rule to get

\[ \frac{y^3}{y^5} = y^{3-5} = y^{-2}. \]
With negative exponents there is no need to state the quotient rule in two parts as we did in Section 4.6. It can be stated simply as
\[
\frac{a^m}{a^n} = a^{m-n}
\]
for any integers \(m\) and \(n\). We list the rules of exponents here for easy reference.

### Rules for Integral Exponents

The following rules hold for nonzero real numbers \(a\) and \(b\) and any integers \(m\) and \(n\).

1. \(a^0 = 1\)  
   **Definition of zero exponent**

2. \(a^m \cdot a^n = a^{m+n}\)  
   **Product rule**

3. \(\frac{a^m}{a^n} = a^{m-n}\)  
   **Quotient rule**

4. \((a^m)^n = a^{mn}\)  
   **Power rule**

5. \((ab)^n = a^n \cdot b^n\)  
   **Power of a product rule**

6. \(\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\)  
   **Power of a quotient rule**

### Example 3

The product and quotient rules for integral exponents

Simplify. Write your answers without negative exponents. Assume that the variables represent nonzero real numbers.

- a) \(b^{-3}b^5\)
- b) \(-3x^{-3} \cdot 5x^2\)
- c) \(\frac{m^6}{m^2}\)
- d) \(\frac{4y^5}{-12y^{-3}}\)

**Solution**

- a) \(b^{-3}b^5 = b^{-3+5} = b^2\)  
  **Product rule**

- b) \(-3x^{-3} \cdot 5x^2 = -15x^{-1}\)  
  **Product rule**

- c) \(\frac{m^6}{m^2} = m^{6-2} = m^4\)  
  **Quotient rule**

- d) \(\frac{4y^5}{-12y^{-3}} = \frac{y^{5-(-3)}}{-3} = -\frac{y^8}{3}\)  
  **Definition of negative exponent**

In the next example we use the power rules with negative exponents.
The power rules for integral exponents

Simplify each expression. Write your answers with positive exponents only. Assume that all variables represent nonzero real numbers.

\( (a^{-3})^2 \) \hspace{1cm} \( (10x^{-3})^{-2} \) \hspace{1cm} \( \left( \frac{4x^{-5}}{y^2} \right)^{-2} \)

**Solution**

a) \( (a^{-3})^2 = a^{-3 \cdot 2} \) \hspace{1cm} Power rule
\[ = a^{-6} \]
\[ = \frac{1}{a^6} \] \hspace{1cm} Definition of negative exponent

b) \( (10x^{-3})^{-2} = 10^{-2}(x^{-3})^{-2} \) \hspace{1cm} Power of a product rule
\[ = 10^{-2}x^{(-3)(-2)} \] \hspace{1cm} Power rule
\[ = x^6 \]
\[ = \frac{x^6}{100} \] \hspace{1cm} Definition of negative exponent

c) \( \left( \frac{4x^{-5}}{y^2} \right)^{-2} = \left( \frac{4x^{-5}}{y^2} \right)^{-2} \) \hspace{1cm} Power of a quotient rule
\[ = \frac{4^{-2}x^{10}}{y^{-4}} \] \hspace{1cm} Power of a product rule and power rule
\[ = 4^{-2} \cdot x^{10} \cdot \frac{1}{y^{-4}} \] \hspace{1cm} Because \( \frac{a}{b} = a \cdot \frac{1}{b} \)
\[ = \frac{1}{4^2} \cdot x^{10} \cdot y^4 \] \hspace{1cm} Definition of negative exponent
\[ = \frac{x^{10}y^4}{16} \] \hspace{1cm} Simplify.

**Converting from Scientific Notation**

Many of the numbers occurring in science are either very large or very small. The speed of light is 983,569,000 feet per second. One millimeter is equal to 0.000001 kilometer. In scientific notation, numbers larger than 10 or smaller than 1 are written by using positive or negative exponents.

Scientific notation is based on multiplication by integral powers of 10. Multiplying a number by a positive power of 10 moves the decimal point to the right:

\[ 10(5.32) = 53.2 \]
\[ 10^2(5.32) = 100(5.32) = 532 \]
\[ 10^3(5.32) = 1000(5.32) = 5320 \]

Multiplying by a negative power of 10 moves the decimal point to the left:

\[ 10^{-1}(5.32) = \frac{1}{10}(5.32) = 0.532 \]
\[ 10^{-2}(5.32) = \frac{1}{100}(5.32) = 0.0532 \]
\[ 10^{-3}(5.32) = \frac{1}{1000}(5.32) = 0.00532 \]
If \( n \) is a positive integer, multiplying by \( 10^n \) moves the decimal point \( n \) places to the right and multiplying by \( 10^{-n} \) moves it \( n \) places to the left.

A number in scientific notation is written as a product of a number between 1 and 10 and a power of 10. For example, \( 3.27 \times 10^6 \) and \( 2.5 \times 10^{-4} \) are numbers in scientific notation. In scientific notation there is one digit to the left of the decimal point.

To convert \( 3.27 \times 10^6 \) to standard notation, move the decimal point six places to the right:

\[
3.27 \times 10^6 = 3,270,000
\]

Of course, it is not necessary to put the decimal point in when writing a whole number.

To convert \( 2.5 \times 10^{-4} \) to standard notation, the decimal point is moved four places to the left:

\[
2.5 \times 10^{-4} = 0.00025
\]

In general, we use the following strategy to convert from scientific notation to standard notation.

**Strategy for Converting from Scientific Notation to Standard Notation**

1. Determine the number of places to move the decimal point by examining the exponent on the 10.
2. Move to the right for a positive exponent and to the left for a negative exponent.

**Example 5**

Converting scientific notation to standard notation

Write in standard notation.

a) \( 7.02 \times 10^6 \)  
   b) \( 8.13 \times 10^{-5} \)

**Solution**

a) Because the exponent is positive, move the decimal point six places to the right:

\[
7.02 \times 10^6 = 7,020,000
\]

b) Because the exponent is negative, move the decimal point five places to the left:

\[
8.13 \times 10^{-5} = 0.0000813
\]

**Converting to Scientific Notation**

To convert a positive number to scientific notation, we just reverse the strategy for converting from scientific notation.

**Strategy for Converting to Scientific Notation**

1. Count the number of places \( (n) \) that the decimal must be moved so that it will follow the first nonzero digit of the number.
2. If the original number was larger than 10, use \( 10^n \).
3. If the original number was smaller than 1, use \( 10^{-n} \).
Remember that the scientific notation for a number larger than 10 will have a positive power of 10 and the scientific notation for a number between 0 and 1 will have a negative power of 10.

**Example 6**

**Converting numbers to scientific notation**

Write in scientific notation.

(a) 7,346,200

(b) 0.0000348

(c) \(135 \times 10^{-12}\)

**Solution**

(a) Because 7,346,200 is larger than 10, the exponent on the 10 will be positive:

\[
7,346,200 = 7.3462 \times 10^6
\]

(b) Because 0.0000348 is smaller than 1, the exponent on the 10 will be negative:

\[
0.0000348 = 3.48 \times 10^{-5}
\]

(c) There should be only one nonzero digit to the left of the decimal point:

\[
135 \times 10^{-12} = 1.35 \times 10^2 \times 10^{-12} = 1.35 \times 10^{-10}
\]

**Computations with Scientific Notation**

An important feature of scientific notation is its use in computations. Numbers in scientific notation are nothing more than exponential expressions, and you have already studied operations with exponential expressions in this section. We use the same rules of exponents on numbers in scientific notation that we use on any other exponential expressions.

**Example 7**

Perform the indicated computations. Write the answers in scientific notation.

(a) \((3 \times 10^6)(2 \times 10^8)\)

(b) \(\frac{4 \times 10^5}{8 \times 10^{-2}}\)

(c) \((5 \times 10^{-7})^3\)

**Solution**

(a) \((3 \times 10^6)(2 \times 10^8) = 3 \cdot 2 \cdot 10^6 \cdot 10^8 = 6 \times 10^{14}\)

(b) \(\frac{4 \times 10^5}{8 \times 10^{-2}} = \frac{4}{8} \cdot \frac{10^5}{10^{-2}} = \frac{1}{2} \cdot 10^{5-(-2)} = 0.5 \times 10^7\)

(c) \((5 \times 10^{-7})^3 = 5^3(10^{-7})^3 = 125 \cdot 10^{-21} = 1.25 \times 10^{2} \cdot 10^{-21} = 1.25 \times 10^{-19}\)
Converting to scientific notation for computations

Perform these computations by first converting each number into scientific notation. Give your answer in scientific notation.

a) \((3,000,000)(0.0002)\)  
b) \((20,000,000)^3(0.0000003)\)

**Solution**

a) \((3,000,000)(0.0002) = 3 \times 10^6 \cdot 2 \times 10^{-4}\) 
   \[= 6 \times 10^2\] 
   Scientific notation, Product rule

b) \((20,000,000)^3(0.0000003) = (2 \times 10^7)^3(3 \times 10^{-7})\) 
   \[= 8 \times 10^{21} \cdot 3 \times 10^{-7}\] 
   \[= 24 \times 10^{14}\] 
   \[= 2.4 \times 10^{14}\] 
   \[24 = 2.4 \times 10^1\] 
   Scientific notation, Power of a product rule, Product rule

**Warm-Ups**

True or false? Explain your answer.

1. \(10^{-2} = \frac{1}{100}\)  
2. \((-\frac{1}{5})^{-1} = 5\)

3. \(3^{-2} \cdot 2^{-1} = 6^{-3}\)  
4. \(\frac{3^{-2}}{3^{-1}} = \frac{1}{3}\)

5. \(23.7 = 2.37 \times 10^{-1}\)  
6. \(0.000036 = 3.6 \times 10^{-5}\)

7. \(25 \cdot 10^7 = 2.5 \times 10^8\)  
8. \(0.442 \times 10^{-3} = 4.42 \times 10^{-4}\)

9. \((3 \times 10^{-9})^2 = 9 \times 10^{-18}\)  
10. \((2 \times 10^{-5})(4 \times 10^4) = 8 \times 10^{-20}\)

@ http://www.napanet.net/education/redwood/departments/mathhandbook/Default.htm#exponents

**Exercises**

**Reading and Writing**  After reading this section, write out the answers to these questions. Use complete sentences.

1. What does a negative exponent mean?

2. What is the correct order for evaluating the operations indicated by a negative exponent?

3. What is the new quotient rule for exponents?

4. How do you convert a number from scientific notation to standard notation?

5. How do you convert a number from standard notation to scientific notation?

6. Which numbers are not usually written in scientific notation?

Variables in all exercises represent positive real numbers. Evaluate each expression. See Example 1.

7. \(3^{-1}\)  
8. \(3^{-3}\)  
9. \((-2)^{-4}\)

10. \((-3)^{-4}\)  
11. \(-4^{-2}\)  
12. \(-2^{-4}\)

13. \(\frac{5^{-2}}{10^{-2}}\)  
14. \(\frac{3^{-4}}{6^{-2}}\)

Simplify. See Example 2.

15. \(\left(\frac{5}{2}\right)^{-3}\)  
16. \(\left(\frac{4}{3}\right)^{-2}\)  
17. \(6^{-1} + 6^{-1}\)

18. \(2^{-1} + 4^{-1}\)  
19. \(\frac{10}{5^3}\)  
20. \(\frac{1}{25 \cdot 10^{-4}}\)

21. \(\frac{1}{4^{-3}} + \frac{3^2}{2^{-1}}\)  
22. \(\frac{2^3}{10^{-2}} - \frac{2}{7^2}\)
Simplify. Write answers without negative exponents. See Example 3.
23. \(x^{-1}x^2\)  
24. \(y^{-3}y^5\)
25. \(-2x^2 \cdot 8x^{-6}\)  
26. \(5y^3(-6y^{-7})\)
27. \(-3a^{-2}(-2a^{-3})\)  
28. \((-b^{-3})(-b^{-5})\)
29. \(\frac{u^{-5}}{u^3}\)  
30. \(\frac{w^{-4}}{w^6}\)
31. \(\frac{8r^{-3}}{-2r^{-5}}\)  
32. \(\frac{-22w^{-4}}{-11w^{-3}}\)
33. \(-6x^5\)  
34. \(\frac{-51y^6}{17y^{-9}}\)

Simplify each expression. Write answers without negative exponents. See Example 4.
35. \((x^2)^{-5}\)  
36. \((y^{-2})^4\)  
37. \((a^{-3})^{-3}\)
38. \((b^{-5})^{-2}\)  
39. \((2x^{-3})^{-4}\)  
40. \((3y^{-1})^{-2}\)
41. \((4x^2y^{-3})^{-2}\)  
42. \((6x^{-2}t)^{-1}\)
43. \(\left(\frac{2x^{-1}}{y^{-3}}\right)^{-2}\)  
44. \(\left(\frac{a^{-2}}{3b^3}\right)^{-3}\)
45. \(\left(\frac{2a^{-3}}{ac^{-2}}\right)^{-4}\)  
46. \(\left(\frac{3w^2}{w^3x}\right)^{-2}\)

Simplify. Write answers without negative exponents.
47. \(2^{-1} \cdot 3^{-1}\)  
48. \(2^{-1} + 3^{-1}\)
49. \((2 \cdot 3^{-1})^{-1}\)  
50. \((2^{-1} + 3)^{-1}\)
51. \((x^{-2})^{-3} + 3x^2(-5x^{-1})\)  
52. \((ab^{-1})^2 - ab(-ab^{-3})\)
53. \(\frac{a^3b^{-2}}{a^{-1}} + \left(\frac{b^{-2}a^3}{b^5}\right)^{-2}\)
54. \(\left(\frac{x^{-3}y^{-1}}{2x}\right)^{-3} + \frac{6x^3y^3}{-3x^{-3}}\)

Write each number in standard notation. See Example 5.
55. \(9.86 \times 10^6\)  
56. \(4.007 \times 10^8\)
57. \(1.37 \times 10^{-3}\)  
58. \(9.3 \times 10^{-5}\)
59. \(1 \times 10^{-6}\)  
60. \(3 \times 10^{-1}\)
61. \(6 \times 10^5\)  
62. \(8 \times 10^6\)

Write each number in scientific notation. See Example 6.
63. \(9000\)
64. \(5,298,000\)
65. \(0.00078\)
66. \(0.000214\)
67. \(0.0000085\)
68. \(5,670,000,000\)
69. \(525 \times 10^9\)
70. \(0.0034 \times 10^{-8}\)

Perform the computations. Write answers in scientific notation. See Example 7.
71. \((3 \times 10^5)(2 \times 10^{-15})\)
72. \((2 \times 10^{-9})(4 \times 10^{23})\)
73. \(4 \times 10^{-8}\)
74. \(2 \times 10^{30}\)
75. \(9 \times 10^{-4}\)
76. \(3 \times 10^{-6}\)
77. \(3 \times 10^{20}\)
78. \(6 \times 10^{-8}\)
79. \(1 \times 10^{-8}\)
80. \((3 \times 10^2)^2\)
81. \((2 \times 10^{-5})^3\)
82. \((5 \times 10^3)^2\)
83. \((5.36 \times 10^4)^{-1}\)
84. \((4 \times 10^{12})^{-1}\)
85. \((6 \times 10^{13})^2\)

Perform the following computations by first converting each number into scientific notation. Write answers in scientific notation. See Example 8.
86. \((4.3)(2,000,000)\)
87. \((4.0000000,000)\)
88. \((4,200,000)(0.00005)\)
89. \((0.00075)(4,000,000)\)
90. \((300)(80,000)\)
91. \((3.56 \times 10^5)(4.43 \times 10^6)\)
92. \(3.4 \times 10^{-8}\)
93. \(2.43 \times 10^{45}\)
94. \((3.56 \times 10^{35})(4.43 \times 10^{56})\)
95. \((8 \times 10^{10}) + (3 \times 10^{20})\)
Solve each problem.

99. **Distance to the sun.** The distance from the earth to the sun is 93 million miles. Express this distance in feet. (1 mile = 5280 feet.)

![Figure for Exercise 99](image)

100. **Speed of light.** The speed of light is $9.83569 \times 10^8$ feet per second. How long does it take light to travel from the sun to the earth? See Exercise 99.

101. **Warp drive, Scotty.** How long does it take a spacecraft traveling at $2 \times 10^3$ miles per hour (warp factor 4) to travel 93 million miles.

102. **Area of a dot.** If the radius of a very small circle is $2.35 \times 10^{-3}$ centimeters, then what is the circle’s area?

103. **Circumference of a circle.** If the circumference of a circle is $5.68 \times 10^9$ feet, then what is its radius?

104. **Diameter of a circle.** If the diameter of a circle is $1.3 \times 10^{-12}$ meters, then what is its radius?

105. **Extracting metals from ore.** Thomas Sherwood studied the relationship between the concentration of a metal in commercial ore and the price of the metal. The accompanying graph shows the Sherwood plot with the locations of several metals marked. Even though the scales on this graph are not typical, the graph can be read in the same manner as other graphs. Note also that a concentration of 100 is 100%.

   a) Use the figure to estimate the price of copper (Cu) and its concentration in commercial ore.

   b) Use the figure to estimate the price of a metal that has a concentration of $10^{-6}$ percent in commercial ore.

   c) Would the four points shown in the graph lie along a straight line if they were plotted in our usual coordinate system?

![Figure for Exercise 105](image)

106. **Recycling metals.** The accompanying graph shows the prices of various metals that are being recycled and the minimum concentration in waste required for recycling. The straight line is the line from the figure for Exercise 105. Points above the line correspond to metals for which it is economically feasible to increase recycling efforts.

   a) Use the figure to estimate the price of mercury (Hg) and the minimum concentration in waste required for recycling mercury.

   b) Use the figure to estimate the price of silver (Ag) and the minimum concentration in waste required for recycling silver.

![Figure for Exercise 106](image)

107. **Present value.** The present value $P$ that will amount to $A$ dollars in $n$ years with interest compounded annually at annual interest rate $r$, is given by

$$ P = A \left(1 + \frac{r}{100}\right)^{-n}. $$

Find the present value that will amount to $50,000 in 20 years at 8% compounded annually.

108. **Investing in stocks.** U.S. small company stocks have returned an average of 14.9% annually for the last 50 years (T. Rowe Price, www.troweprice.com). Use the present value formula from the previous exercise to find the amount invested today in small company stocks that would be worth $1 million in 50 years, assuming that small company stocks continue to return 14.9% annually for the next 50 years.

![Figure for Exercise 108](image)
**Getting More Involved**

109. **Exploration.**
   a) If \( w^{-3} < 0 \), then what can you say about \( w \)?
   b) If \((-5)^m < 0\), then what can you say about \( m \)?
   c) What restriction must be placed on \( w \) and \( m \) so that \( w^m < 0 \)?

**Collaborative Activities**

**Area as a Model of FOIL**

Sometimes we can use drawings to represent mathematical operations. The area of a rectangle can represent the process we use when multiplying binomials. The rectangle below represents the multiplication of the binomials \((x + 3)\) and \((x + 5)\):

\[
\begin{array}{c|c|c}
   & 3x & 15 \\
\hline
x^2 & 5x & \\
\hline
x & 5 & \\
\hline
\end{array}
\]

The areas of the inner rectangles are \(x^2, 3x, 5x, \) and 15.

The area of the red rectangle equals the sum of the areas of the four inner rectangles.

Area of red rectangle:

\[
(x + 3)(x + 5) = x^2 + 3x + 5x + 15 = x^2 + 8x + 15
\]

1. **a.** With your partner, find the areas of the inner rectangles to find the product \((x + 2)(x + 7)\) below:

\[
\begin{array}{c|c|c}
   & 2 & \\
\hline
x & 7 & \\
\hline
\end{array}
\]

\[
(x + 2)(x + 7) = ?
\]

**b.** Find the same product \((x + 2)(x + 7)\) using FOIL.

For problem 2, student A uses FOIL to find the given product while student B finds the area with the diagram.

2. \((x + 8)(x + 1) = ?\)

**Thinking in reverse:** Work together to complete the product that is represented by the given diagram.

3. \((x + 5)(x + 4) = ?\)

4. Student A draws a diagram to find the product \((x + 3)(x + 7)\). Student B finds \((x + 3)(x + 7)\) using FOIL.

5. Student B draws a diagram to find the product \((x + 2)(x + 1)\). Student A finds \((x + 2)(x + 1)\) using FOIL.

**Extension:** Make up a FOIL problem, then have your partner draw a diagram of it.

110. **Discussion.** Which of the following expressions is not equal to \(-1\)? Explain your answer.

   a) \(-1^{-1}\)
   b) \(-1^{-2}\)
   c) \((-1^{-1})^{-1}\)
   d) \((-1)^{-1}\)
   e) \((-1)^{-2}\)
## WRAP-UP

### SUMMARY

#### Polynomials

<table>
<thead>
<tr>
<th>Term</th>
<th>A number or the product of a number and one or more variables raised to powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example</td>
<td>$5x^3, -4x, 7$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>A single term or a finite sum of terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example</td>
<td>$2x^5 - 9x^2 + 11$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degree of a polynomial</th>
<th>The highest degree of any of the terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example</td>
<td>Degree of $2x - 9$ is 1. Degree of $5x^3 - x^2$ is 3.</td>
</tr>
</tbody>
</table>

#### Adding, Subtracting, and Multiplying Polynomials

**Add or subtract** Add or subtract the like terms.

<table>
<thead>
<tr>
<th>$\text{(x + 1)} + (x - 4)$</th>
<th>$2x - 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x^2 - 3x) - (4x^2 - x)$</td>
<td>$-3x^2 - 2x$</td>
</tr>
</tbody>
</table>

**Multiply monomials** Use the product rule for exponents.

| $-2x^5 \cdot 6x^8$ | $-12x^{13}$ |

**Multiply polynomials** Multiply each term of one polynomial by every term of the other polynomial, then combine like terms.

<table>
<thead>
<tr>
<th>$x^2 + 2x + 5$</th>
<th>$x - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-x^2 - 2x - 5$</td>
<td>$x^3 + 2x^2 + 5x$</td>
</tr>
<tr>
<td>$x^3 + x^2 + 3x - 5$</td>
<td>$x^3 + x^2 + 3x - 5$</td>
</tr>
</tbody>
</table>

**Binomials**

**FOIL** A method for multiplying two binomials quickly

| $(x - 2)(x + 3)$ | $x^2 + x - 6$ |

**Square of a sum** $(a + b)^2 = a^2 + 2ab + b^2$

| $(x + 3)^2$ | $x^2 + 6x + 9$ |

**Square of a difference** $(a - b)^2 = a^2 - 2ab + b^2$

| $(m - 5)^2$ | $m^2 - 10m + 25$ |

**Product of a sum and a difference** $(a - b)(a + b) = a^2 - b^2$

| $(x + 2)(x - 2)$ | $x^2 - 4$ |

#### Dividing Polynomials

**Dividing monomials** Use the quotient rule for exponents

| $8x^5 \div (2x^2)$ | $4x^3$ |

**Divide a polynomial by a monomial** Divide each term of the polynomial by the monomial.

| $\frac{3x^5 + 9x}{3x}$ | $x^4 + 3$ |

**Divide a polynomial by a binomial** If the divisor is a binomial, use long division.

| $x - 7 \leftarrow \text{Quotient}$ |
| $\text{Divisor} \rightarrow x + 2 \sqrt{x^2 - 5x - 4} \leftarrow \text{Dividend}$ |
| $\frac{-7x - 4}{\sqrt{x^2 + 2x}}$ |
| $-7x - 14$ |
| $10 \leftarrow \text{Remainder}$ |
### Rules of Exponents

The following rules hold for any integers \( m \) and \( n \), and nonzero real numbers \( a \) and \( b \).

<table>
<thead>
<tr>
<th>Rule</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero exponent ( a^0 = 1 )</td>
<td>( 2^0 = 1, (-34)^0 = 1 )</td>
</tr>
<tr>
<td>Product rule ( a^m \cdot a^n = a^{m+n} )</td>
<td>( a^2 \cdot a^3 = a^5 )</td>
</tr>
<tr>
<td>Quotient rule ( \frac{a^m}{a^n} = a^{m-n} )</td>
<td>( x^8 \div x^2 = x^6, \frac{y^3}{y^0} = y^3 = 1 )</td>
</tr>
<tr>
<td>Power rule ( (a^m)^n = a^{mn} )</td>
<td>( (2^2)^3 = 2^6, (w^5)^3 = w^{15} )</td>
</tr>
<tr>
<td>Power of a product rule ( (ab)^n = a^n b^n )</td>
<td>( (2t)^3 = 8t^3 )</td>
</tr>
<tr>
<td>Power of a quotient rule ( \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} )</td>
<td>( \left(\frac{x}{3}\right)^3 = \frac{x^3}{27} )</td>
</tr>
</tbody>
</table>

### Negative Exponents

If \( n \) is a positive integer and \( a \) is a nonzero real number, then \( a^{-n} = \frac{1}{a^n} \).

<table>
<thead>
<tr>
<th>Rule</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative integral exponents</td>
<td>( 3^{-2} = \frac{1}{3^2}, x^{-5} = \frac{1}{x^5} )</td>
</tr>
<tr>
<td>Rules for negative exponents</td>
<td>( \left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3, 5^{-1} = \frac{1}{5} )</td>
</tr>
<tr>
<td>( a^{-n} = \left(\frac{1}{a}\right)^n )</td>
<td>( \frac{1}{w^{-8}} = w^8 )</td>
</tr>
</tbody>
</table>

### Scientific Notation

**Converting from scientific notation**

1. Find the number of places to move the decimal point by examining the exponent on the 10.
2. Move to the right for a positive exponent and to the left for a negative exponent.

**Examples**

\( 5.6 \times 10^3 = 5600 \)
\( 9 \times 10^{-4} = 0.0009 \)

**Converting into scientific notation (positive numbers)**

1. Count the number of places \( (n) \) that the decimal point must be moved so that it will follow the first nonzero digit of the number.
2. If the original number was larger than 10, use \( 10^n \).
3. If the original number was smaller than 1, use \( 10^{-n} \).

**Examples**

\( 304.6 = 3.046 \times 10^2 \)
\( 0.0035 = 3.5 \times 10^{-3} \)
For each mathematical term, choose the correct meaning.

1. **term**
   a. an expression containing a number or the product of a number and one or more variables
   b. the amount of time spent in this course
   c. a word that describes a number
   d. a variable

2. **polynomial**
   a. four or more terms
   b. many numbers
   c. a sum of four or more numbers
   d. a single term or a finite sum of terms

3. **degree of a polynomial**
   a. the number of terms in a polynomial
   b. the highest degree of any of the terms of a polynomial
   c. the coefficient of the first term when a polynomial is written with decreasing exponents
   d. the most important coefficient

4. **leading coefficient**
   a. the first coefficient
   b. the largest coefficient
   c. the coefficient of the first term when a polynomial is written with decreasing exponents
   d. the most important coefficient

5. **monomial**
   a. a single polynomial
   b. one number
   c. an equation that has only one solution
   d. a polynomial that has one term

6. **FOIL**
   a. a method for adding polynomials
   b. first, outer, inner, last
   c. an equation with no solution
   d. a polynomial with five terms

7. **dividend**
   a. \( a \) in \( a/b \)
   b. \( b \) in \( a/b \)
   c. the result of \( a/b \)
   d. what a bank pays on deposits

8. **divisor**
   a. \( a \) in \( a/b \)
   b. \( b \) in \( a/b \)
   c. \( a/b \)
   d. the divisor plus the remainder

9. **quotient**
   a. \( a \) in \( a/b \)
   b. \( b \) in \( a/b \)
   c. \( a/b \)
   d. two visors

10. **binomial**
    a. a polynomial with two terms
    b. any two numbers
    c. the two coordinates in an ordered pair
    d. an equation with two variables

11. **integral exponent**
    a. an exponent that is an integer
    b. a positive exponent
    c. a rational exponent
    d. a fractional exponent

12. **scientific notation**
    a. the notation of rational exponents
    b. the notation of algebra
    c. a notation for expressing large or small numbers with powers of 10
    d. radical notation

### REVIEW EXERCISES

#### 4.1 Perform the indicated operations.

1. \((2w - 6) + (3w + 4)\)
2. \((1 - 3y) + (4y - 6)\)
3. \((x^2 - 2x - 5) - (x^2 + 4x - 9)\)
4. \((3 - 5x - x^2) - (x^3 - 7x + 8)\)
5. \((5 - 3w + w^2) + (w^2 - 4w - 9)\)
6. \((-2r^2 + 3t - 4) + (r^2 - 7t + 2)\)
7. \((4 - 3m - m^2) - (m^2 - 6m + 5)\)
8. \((n^3 - n^2 + 9) - (n^4 - n^3 + 5)\)

#### 4.2 Perform the indicated operations.

9. \(5x^2 \cdot (-10x^3)\)
10. \(3h^3 t^5 \cdot 2h^2 t^5\)
11. \((-11a^2)^3\)
12. \((12b)^3\)
13. \(x - 5(x - 3)\)
14. \(x - 4(x - 9)\)
15. \(5x + 3(x^2 - 5x + 4)\)
16. \(5 + 4x^2(x - 5)\)
17. \(3m^2(5m^3 - m + 2)\)
18. \(-4a^4(a^2 + 2a + 4)\)

**Study Tip**

Note how the review exercises are arranged according to the sections in this chapter. If you are having trouble with a certain type of problem, refer back to the appropriate section for examples and explanations.
19. \((x - 5)(x^2 - 2x + 10)\)  
20. \((x + 2)(x^2 - 2x + 4)\)
21. \((x^2 - 2x + 4)(3x - 2)\)  
22. \((5x + 3)(x^2 - 5x + 4)\)

4.3 Perform the indicated operations.
23. \((q - 6)(q + 8)\)  
24. \((w + 5)(w + 12)\)
25. \((2t - 3)(t - 9)\)  
26. \((5r + 1)(5r + 2)\)
27. \((4y - 3)(5y + 2)\)  
28. \((11y + 1)(y + 2)\)
29. \((3x^2 + 5)(2x^2 + 1)\)  
30. \((x^3 - 7)(2x^3 + 7)\)

4.4 Perform the indicated operations. Try to write only the answers.
31. \((z - 7)(z + 7)\)  
32. \((a - 4)(a + 4)\)
33. \((y + 7)^2\)  
34. \((a + 5)^2\)
35. \((w - 3)^2\)  
36. \((a - 6)^2\)
37. \((x^2 - 3)(x^2 + 3)\)  
38. \((2b^2 - 1)(2b^2 + 1)\)
39. \((3a + 1)^2\)  
40. \((1 - 3c)^2\)
41. \((4 - y)^2\)  
42. \((9 - r)^2\)

4.5 In Exercises 43–54, find each quotient.
43. \(-10x^5 \div (2x^3)\)  
44. \(-6x^3y^2 \div (-2x^2y^3)\)
45. \(-3a^3b^4d^6 \div (3h^2r^2)\)  
46. \(3x - 9 \div -3\)
47. \(9x^3 - 6x^2 + 3x \div -3x\)
48. \(-8x^3y^3 + 4x^3y^4 - 2xy^3 \div 2xy^2\)
51. \((a - 1) \div (1 - a)\)  
52. \((t - 3) \div (3 - t)\)
53. \((m^4 - 16) \div (m - 2)\)  
54. \((x^4 - 1) \div (x - 1)\)

Find the quotient and remainder.
55. \((3m^3 - 9m^2 + 18m) \div (3m)\)  
56. \((8x^3 - 4x^2 - 18x) \div (2x)\)
57. \((b^3 - 3b + 5) \div (b + 2)\)  
58. \((r^2 - 5r + 9) \div (r - 3)\)
59. \((4x^2 - 9) \div (2x + 1)\)  
60. \((9y^3 + 2y) \div (3y + 2)\)

61. \((x^3 + x^2 - 11x + 10) \div (x - 1)\)
62. \((y^3 - 9y^2 + 3y - 6) \div (y + 1)\)

Write each expression in the form \(\text{quotient} + \frac{\text{remainder}}{\text{divisor}}\).
63. \(\frac{2x}{x - 3}\)  
64. \(\frac{3x}{x - 4}\)
65. \(\frac{2x}{1 - x}\)  
66. \(\frac{3x}{5 - x}\)
67. \(\frac{x^2 - 3}{x + 1}\)  
68. \(\frac{x^2 + 3x + 1}{x - 3}\)
69. \(\frac{x^2}{x + 1}\)  
70. \(-\frac{2x^2}{x - 3}\)

4.6 Simplify each expression.
71. \(2y^{10} \cdot 3y^{20}\)  
72. \((-3a^5)(5a^3)\)
73. \(-10bc^3d^6 \div 2b^5 c^9\)  
74. \(-30k^3y^9 \div 15k^3y^2\)
75. \((b^3)^6\)  
76. \((y^5)^8\)
77. \((-2x^3y^2)^3\)  
78. \((-3a^4b^6)^4\)
79. \(\left(\frac{2a^3}{b}\right)^3\)  
80. \(\left(\frac{3y^3}{2}\right)^3\)
81. \(\left(\frac{-6x^3y^3}{-3z^6}\right)^3\)  
82. \(\left(-3a^4b^6\right)^4 \div 6a^6b^{12}\)

For the following exercises, assume that all of the variables represent positive real numbers.

4.7 Simplify each expression. Use only positive exponents in answers.
83. \(2^{-5}\)  
84. \(-2^{-4}\)  
85. \(10^{-3}\)
86. \(5^{-1} \cdot 5^{0}\)  
87. \(x^5x^{-8}\)  
88. \(a^{-3}a^{-9}\)
89. \(\frac{a^{-8}}{a^{-12}}\)  
90. \(\frac{a^{10}}{a^4}\)  
91. \(\frac{a^3}{a^{-7}}\)
92. \(\frac{b^{-2}}{b^{-6}}\)  
93. \((x^{-3})^4\)  
94. \((x^5)^{-10}\)
95. \((2x^{-2})^{-3}\)  
96. \((3y^{-5})^2\)  
97. \(\left(\frac{a}{3b^3}\right)^{-2}\)
98. \(\left(\frac{a^{-2}}{5b}\right)^{-3}\)
Convert each number in scientific notation to a number in standard notation, and convert each number in standard notation to a number in scientific notation.

99. 5000 100. 0.00009

101. $3.4 \times 10^5$ 102. $5.7 \times 10^{-8}$

103. 0.0000461 104. 44,000

105. $5.69 \times 10^{-6}$ 106. $5.5 \times 10^9$

Perform each computation without using a calculator. Write answers in scientific notation.

107. $(3.5 \times 10^9)(2.0 \times 10^{-12})$

108. $(9 \times 10^{15})(2 \times 10^{17})$

109. $(2 \times 10^{-4})^4$

110. $(-3 \times 10^3)^3$

111. $(0.00000004)(2,000,000,000)$

112. $(3,000,000,000) \div (0.000002)$

113. $(0.000002)^5$

114. $(50,000,000,000)^3$

**MISCELLANEOUS**

Perform the indicated operations.

115. $(x + 3)(x + 7)$ 116. $(k + 5)(k + 4)$

117. $(t - 3y)(t - 4y)$ 118. $(t + 7z)(t + 6z)$

119. $(2x^3)^0 + (2y)^0$

120. $(4y^2 - 9)^0$

121. $(-3hr)^3$

122. $(-9y^4x^5)^2$

123. $(2w + 3)(w - 6)$ 124. $(3x + 5)(2x - 6)$

125. $(3u - 5v)(3u + 5v)$ 126. $(9x^2 - 2)(9x^2 + 2)$

127. $(3h + 5)^2$

128. $(4v - 3)^2$

129. $(x + 3)^3$

130. $(k - 10)^3$

131. $(-7st)(-2s^3t^2)$

132. $-5w^3r^2 \cdot 2w^4r^8$

133. $\left(\frac{k^4m^2}{2k^2m^2}\right)^4$

134. $\left(\frac{-6h^3y^5}{2h^2y^2}\right)^4$

135. $(5x^2 - 8x - 8) - (4x^2 + x - 3)$

136. $(4x^2 - 6x - 8) - (9x^2 - 5x + 7)$

137. $(2x^2 - 2x - 3) + (3x^2 + x - 9)$

138. $(x^2 - 3x - 1) + (x^2 - 2x + 1)$

139. $(x + 4)(x^2 - 5x + 1)$

140. $(2x^2 - 7x + 4)(x + 3)$

141. $(x^2 + 4x - 12) \div (x - 2)$

142. $(a^2 - 3a - 10) \div (a - 5)$

Solve each problem.

143. **Roundball court.** The length of a basketball court is 44 feet larger than its width. Find polynomials that represent its perimeter and area. The actual width of a basketball court is 50 feet. Evaluate these polynomials to find the actual perimeter and area of the court.

144. **Badminton court.** The width of a badminton court is 24 feet less than its length. Find polynomials that represent its perimeter and area. The actual length of a badminton court is 44 feet. Evaluate these polynomials to find the perimeter and area of the court.

145. **Smoke alert.** A retailer of smoke alarms knows that at a price of $p$ dollars each, she can sell $600 - 15p$ smoke alarms per week. Find a polynomial that represents the weekly revenue for the smoke alarms. Find the revenue for a week in which the price is $12 per smoke alarm. Use the bar graph to find the price per smoke alarm that gives the maximum weekly revenue.

146. **Boom box sales.** A retailer of boom boxes knows that at a price of $q$ dollars each, he can sell $900 - 3q$ boom boxes per month. Find a polynomial that represents the monthly revenue for the boom boxes? How many boom boxes will he sell if the price is $300 each?
CHAPTER 4 TEST

Perform the indicated operations.
1. \((7x^3 - x^2 - 6) + (5x^2 + 2x - 5)\)
2. \((x^2 - 3x - 5) - (2x^2 + 6x - 7)\)
3. \(\frac{6y^3 - 9y^2}{-3y}\)
4. \((x - 2) \div (2 - x)\)
5. \((x^3 - 2x^2 - 4x + 3) \div (x - 3)\)
6. \(3x^2(5x^3 - 7x^2 + 4x - 1)\)

Find the products.
7. \((x + 5)(x - 2)\)
8. \((3a - 7)(2a + 5)\)
9. \((a - 7)^2\)
10. \((4x + 3y)^2\)
11. \((b - 3)(b + 3)\)
12. \((3t^2 - 7)(3t^2 + 7)\)
13. \((4x^2 - 3)(x^2 + 2)\)
14. \((x - 2)(x + 3)(x - 4)\)

Write each expression in the form \(\frac{\text{quotient} + \text{remainder}}{\text{divisor}}\).
15. \(\frac{2x}{x - 3}\)
16. \(\frac{x^2 - 3x + 5}{x + 2}\)

Use the rules of exponents to simplify each expression. Write answers without negative exponents.
17. \(-5x^3 \cdot 7x^5\)
18. \(3x^3y \cdot (2x)^4\)
19. \(-4a^6b^5 \div (2a^5b)\)
20. \(3x^{-2} \cdot 5x^2\)
21. \(\left(\frac{-2a}{b^5}\right)^5\)
22. \(-6a^7b^5c^3\)
23. \(\frac{6t^{-7}}{2t^8}\)
24. \(\frac{w^{-6}}{w^{-4}}\)
25. \((-3s^{-3}t^2)^{-2}\)
26. \((-2x^{-6}y)^3\)

Convert to scientific notation.
27. \(5,433,000\)
28. \(0.0000065\)

Perform each computation by converting to scientific notation. Give answers in scientific notation.
29. \((80,000)(0.000006)\)
30. \((0.0000003)^4\)

Solve each problem.
31. Find the quotient and remainder when \(x^2 - 5x + 9\) is divided by \(x - 3\).
32. Subtract \(3x^2 - 4x - 9\) from \(x^2 - 3x + 6\).
33. The width of a pool table is \(x\) feet, and the length is 4 feet longer than the width. Find polynomials that represent the area and the perimeter of the pool table. Evaluate these polynomials for a width of 4 feet.
34. If a manufacturer charges \(q\) dollars each for footballs, then he can sell \(3000 - 150q\) footballs per week. Find a polynomial that represents the revenue for one week. Find the weekly revenue if the price is $8 for each football.
Simplify each expression.

1. \(-16 \div (-2)\)
2. \((-2)^3 - 1\)
3. \((-5)^2 - 3(-5) + 1\)
4. \(2^{10} \cdot 2^{15}\)
5. \(2^{15} \div 2^{10}\)
6. \(2^{10} - 2^{5}\)
7. \(3^2 \cdot 4^2\)
8. \((172 - 85) \div (85 - 172)\)
9. \((5 + 3)^2\)
10. \(5^2 + 3^2\)
11. \((30 - 1)(30 + 1)\)
12. \((30 + 1)^2\)

Perform the indicated operations.

13. \((x + 3)(x + 5)\)
14. \((x^2 + 8x + 15) \div (x + 5)\)
15. \(x + 3(x + 5)\)
16. \((x^2 + 8x + 15)(x + 5)\)
17. \(-5t^3v \cdot 3t^2v^6\)
18. \((-10t^3v^5) \div (-2t^2v)\)
19. \((-6y^3 + 8y^2) \div (-2y^2)\)
20. \((y^2 - 3y - 9) - (-3y^2 + 2y - 6)\)

Solve each equation.

21. \(2x + 1 = 0\)
22. \(x - 7 = 0\)
23. \(2x - 3 = 0\)
24. \(3x - 7 = 5\)
25. \(8 - 3x = x + 20\)
26. \(4 - 3(x + 2) = 0\)

Solve the problem.

27. **Average cost.** Pineapple Recording plans to spend $100,000 to record a new CD by the Woozies and $2.25 per CD to manufacture the disks. The polynomial \(2.25n + 100,000\) represents the total cost in dollars for recording and manufacturing \(n\) disks. Find an expression that represents the average cost per disk by dividing the total cost by \(n\). Find the average cost per disk for \(n = 1000, 100,000,\) and \(1,000,000\). What happens to the large initial investment of $100,000 if the company sells one million CDs?