5.3 Systems of Linear Equations in Three Variables

OBJECTIVES

1. Find ordered triples associated with three equations
2. Solve a system by the addition method
3. Interpret a solution graphically
4. Use a system of three equations to solve an application

Suppose an application involves three quantities that we want to label \(x, y,\) and \(z\). A typical equation used in solving the application might be

\[2x + 4y - z = 8\]

This is called a linear equation in three variables. The solution for such an equation is an ordered triple \((x, y, z)\) of real numbers that satisfies the equation. For example, the ordered triple \((2, 1, 0)\) is a solution for the equation above because substituting 2 for \(x\), 1 for \(y\), and 0 for \(z\) results in the following true statement.

\[2 \cdot 2 + 4 \cdot 1 - 0 \neq 8\]
\[4 + 4 \neq 8\]
\[8 = 8 \quad \text{True}\]

Of course, other solutions, in fact infinitely many, exist. You might want to verify that \((1, 1, -2)\) and \((3, 1, 2)\) are also solutions. To extend the concepts of the last section, we want to consider systems of three linear equations in three variables such as

\[x + y + z = 5\]
\[2x - y + z = 9\]
\[x - 2y + 3z = 16\]

The solution for such a system is the set of all ordered triples that satisfy each equation of the system. In this case, you can verify that \((2, -1, 4)\) is a solution for the system because that ordered triple makes each equation a true statement.

Let’s turn now to the process of solving such a system. In this section, we will consider the addition method. We will then apply what we have learned to solving applications.

The central idea is to choose two pairs of equations from the system and, by the addition method, to eliminate the same variable from each of those pairs. The method is best illustrated by example. So let’s proceed to see how the solution for the previous system was determined.

Example 1

Solving a Linear System in Three Variables

Solve the system.

\[x + y + z = 5 \quad (1)\]
\[2x - y + z = 9 \quad (2)\]
\[x - 2y + 3z = 16 \quad (3)\]
First we choose two of the equations and the variable to eliminate. Variable $y$ seems convenient in this case. Pairing equations (1) and (2) and then adding, we have

\[
\begin{align*}
\begin{align*}
\begin{align*}
&x + y + z = 5 \\
&2x - y + z = 9 \\
&x + 2z = 14
\end{align*}
\end{align*}
\end{align*}
\]  

(4)

We now want to choose a different pair of equations to eliminate $y$. Using equations (1) and (3) this time, we multiply equation (1) by 2 and then add the result to equation (3):

\[
\begin{align*}
\begin{align*}
\begin{align*}
&2x + 2y + 2z = 10 \\
&x - 2y + 3z = 16 \\
&3x + 5z = 26
\end{align*}
\end{align*}
\end{align*}
\]  

(5)

We now have equations (4) and (5) in variables $x$ and $z$.

\[
\begin{align*}
&3x + 2z = 14 \\
&3x + 5z = 26
\end{align*}
\]

Because we are now dealing with a system of two equations in two variables, any of the methods of the previous section apply. We have chosen to multiply equation (4) by $-1$ and then add that result to equation (5). This yields

\[
\begin{align*}
&3z = 12 \\
&z = 4
\end{align*}
\]

Substituting $z = 4$ in equation (4) gives

\[
\begin{align*}
&3x + 2 \cdot 4 = 14 \\
&3x + 8 = 14 \\
&3x = 6 \\
&x = 2
\end{align*}
\]

Finally, letting $x = 2$ and $z = 4$ in equation (1) gives

\[
\begin{align*}
&2 + y + 4 = 5 \\
&y = -1
\end{align*}
\]

and $(2, -1, 4)$ is shown to be the solution for the system.

**Check Yourself 1**

**Solve the system.**

\[
\begin{align*}
&x - 2y + z = 0 \\
&2x + 3y - z = 16 \\
&3x - y - 3z = 23
\end{align*}
\]

One or more of the equations of a system may already have a missing variable. The elimination process is simplified in that case, as Example 2 illustrates.
Example 2

Solving a Linear System in Three Variables

Solve the system.

\begin{align*}
2x + y - z &= -3 \\
y + z &= 2 \\
4x - y + z &= 12
\end{align*}

Noting that equation (7) involves only \(y\) and \(z\), we must simply find another equation in those same two variables. Multiply equation (6) by \(-2\) and add the result to equation (8) to eliminate \(x\).

\begin{align*}
-4x - 2y + 2z &= 6 \\
4x - y + z &= 12 \\
-3y + 3z &= 18 \\
y - z &= -6
\end{align*}

We now form a system consisting of equations (7) and (9) and solve as before.

\begin{align*}
y + z &= 2 \\
y - z &= -6 & \text{Adding eliminates } z. \\
2y &= -4 \\
y &= -2 \\
\end{align*}

From equation (7), if \(y = -2\),

\begin{align*}
-2 + z &= 2 \\
z &= 4
\end{align*}

and from equation (6), if \(y = -2\) and \(z = 4\),

\begin{align*}
2x - 2 - 4 &= -3 \\
2x &= 3 \\
x &= \frac{3}{2}
\end{align*}

The solution for the system is

\[\left(\frac{3}{2}, -2, 4\right)\]

**CHECK YOURSELF 2**

Solve the system.

\begin{align*}
x + 2y - z &= -3 \\
x - y + z &= 2 \\
x - z &= 3
\end{align*}
The following algorithm summarizes the procedure for finding the solutions for a linear system of three equations in three variables.

**Step by Step: Solving a System of Three Equations in Three Unknowns**

**Step 1** Choose a pair of equations from the system, and use the addition method to eliminate one of the variables.

**Step 2** Choose a different pair of equations, and eliminate the same variable.

**Step 3** Solve the system of two equations in two variables determined in steps 1 and 2.

**Step 4** Substitute the values found above into one of the original equations, and solve for the remaining variable.

**Step 5** The solution is the ordered triple of values found in steps 3 and 4. It can be checked by substituting into the other equations of the original system.

Systems of three equations in three variables may have (1) exactly one solution, (2) infinitely many solutions, or (3) no solution. Before we look at an algebraic approach in the second and third cases, let’s discuss the geometry involved.

The graph of a linear equation in three variables is a plane (a flat surface) in three dimensions. Two distinct planes either will be parallel or will intersect in a line.

If three distinct planes intersect, that intersection will be either a single point (as in our first example) or a line (think of three pages in an open book—they intersect along the binding of the book).

Let’s look at an example of how we might proceed in these cases.

**Example 3**

**Solving a Dependent Linear System in Three Variables**

Solve the system.

\[
\begin{align*}
x + 2y - z &= 5 \quad (10) \\
x - y + z &= -2 \quad (11) \\
-5x - 4y + z &= -11 \quad (12)
\end{align*}
\]

We begin as before by choosing two pairs of equations from the system and eliminating the same variable from each of the pairs. Adding equations (10) and (11) gives

\[
2x + y = 3 \quad (13)
\]

Adding equations (10) and (12) gives

\[
-4x - 2y = -6 \quad (14)
\]

Now consider the system formed by equations (13) and (14). We multiply equation (13) by 2 and add again:

\[
\begin{align*}
4x + 2y &= 6 \\
-4x - 2y &= -6 \\
0 &= 0
\end{align*}
\]
This true statement tells us that the system has an infinite number of solutions (lying along a straight line). Again, such a system is dependent.

**CHECK YOURSELF 3**

Solve the system.

\[
\begin{align*}
2x - y + 3z &= 3 \\
-x + y - 2z &= 1 \\
y - z &= 5
\end{align*}
\]

There is a third possibility for the solutions of systems in three variables, as Example 4 illustrates.

**Example 4**

Solving an Inconsistent Linear System in Three Variables

Solve the system.

\[
\begin{align*}
3x + y - 3z &= 1 & (15) \\
-2x - y + 2z &= 1 & (16) \\
-x - y + z &= 2 & (17)
\end{align*}
\]

This time we eliminate variable \( y \). Adding equations (15) and (16), we have

\[x - z = 2 \quad (18)\]

Adding equations (15) and (17) gives

\[2x - 2z = 3 \quad (19)\]

Now, multiply equation (18) by \(-2\) and add the result to equation (19).

\[
\begin{align*}
-2x + 2z &= -4 \\
2x - 2z &= 3 \\
0 &= -1
\end{align*}
\]

All the variables have been eliminated, and we have arrived at a contradiction, \(0 = -1\). This means that the system is **inconsistent** and has no solutions. There is no point common to all three planes.

**CHECK YOURSELF 4**

Solve the system.

\[
\begin{align*}
x - y - z &= 0 \\
-3x + 2y + z &= 1 \\
3x - y + z &= -1
\end{align*}
\]
As a closing note, we have by no means illustrated all possible types of inconsistent and dependent systems. Other possibilities involve either distinct parallel planes or planes that coincide. The solution techniques in these additional cases are, however, similar to those illustrated above.

In many instances, if an application involves three unknown quantities, you will find it useful to assign three variables to those quantities and then build a system of three equations from the given relationships in the problem. The extension of our problem-solving strategy is natural, as Example 5 illustrates.

### Example 5

#### Solving a Number Problem

The sum of the digits of a three-digit number is 12. The tens digit is 2 less than the hundreds digit, and the units digit is 4 less than the sum of the other two digits. What is the number?

**Step 1** The three unknowns are, of course, the three digits of the number.

**Step 2** We now want to assign variables to each of the three digits. Let $u$ be the units digit, $t$ be the tens digit, and $h$ be the hundreds digit.

**Step 3** There are three conditions given in the problem that allow us to write the necessary three equations. From those conditions

\[
\begin{align*}
    h + t + u &= 12 \\
    t &= h - 2 \\
    u &= h + t - 4
\end{align*}
\]

**Step 4** There are various ways to approach the solution. To use addition, write the system in the equivalent form

\[
\begin{align*}
    h + t + u &= 12 \\
    -h + t &= -2 \\
    -h - t + u &= -4
\end{align*}
\]

and solve by our earlier methods. The solution, which you can verify, is $h = 5$, $t = 3$, and $u = 4$. The desired number is 534.

**Step 5** To check, you should show that the digits of 534 meet each of the conditions of the original problem.

#### Check Yourself 5

The sum of the measures of the angles of a triangle is $180^\circ$. In a given triangle, the measure of the second angle is twice the measure of the first. The measure of the third angle is $30^\circ$ less than the sum of the measures of the first two. Find the measure of each angle.

Let’s continue with a financial application that will lead to a system of three equations.
Example 6

Solving an Investment Application

Monica decided to divide a total of $42,000 into three investments: a savings account paying 5% interest, a time deposit paying 7%, and a bond paying 9%. Her total annual interest from the three investments was $2600, and the interest from the savings account was $200 less than the total interest from the other two investments. How much did she invest at each rate?

Step 1 The three amounts are the unknowns.

Step 2 We let $s$ be the amount invested at 5%, $t$ the amount at 7%, and $b$ the amount at 9%. Note that the interest from the savings account is then $0.05s$, and so on.

A table will help with the next step.

<table>
<thead>
<tr>
<th></th>
<th>5%</th>
<th>7%</th>
<th>9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Principal</td>
<td>$s$</td>
<td>$t$</td>
<td>$b$</td>
</tr>
<tr>
<td>Interest</td>
<td>$0.05s$</td>
<td>$0.07t$</td>
<td>$0.09b$</td>
</tr>
</tbody>
</table>

Step 3 Again there are three conditions in the given problem. By using the table above, they lead to the following equations.

\[ s + t + b = 42,000 \]
\[ 0.05s + 0.07t + 0.09b = 2600 \]
\[ 0.05s = 0.07t + 0.09b - 200 \]

Step 4 We clear of decimals and solve as before, with the result

\[ s = 24,000 \quad t = 11,000 \quad b = 7000 \]

Step 5 We leave the check of these solutions to you.

NOTE Again, we choose letters that suggest the unknown quantities—$s$ for savings, $t$ for time deposit, and $b$ for bond.

NOTE For 1 year, the interest formula is

\[ I = Pr \]

(interest equals principal times rate).

NOTE Total invested.

NOTE Total interest.

NOTE The savings interest was $200 less than that from the other two investments.

NOTE Find the interest earned from each investment, and verify that the conditions of the problem are satisfied.

CHECK YOURSELF 6

Glenn has a total of $11,600 invested in three accounts: a savings account paying 6% interest, a stock paying 8%, and a mutual fund paying 10%. The annual interest from the stock and mutual fund is twice that from the savings account, and the mutual fund returned $120 more than the stock. How much did Glenn invest in each account?
CHECK YOURSELF ANSWERS

1. $(5, 1, -3)$  
2. $(1, -3, -2)$  
3. The system is dependent (there are an infinite number of solutions).  
4. The system is inconsistent (there are no solutions).  
5. The three angles are $35^\circ$, $70^\circ$, and $75^\circ$.  
6. $5000$ in savings, $3000$ in stocks, and $3600$ in mutual funds.
5.3 Exercises

In exercises 1 to 20, solve each system of equations. If a unique solution does not exist, state whether the system is inconsistent or has an infinite number of solutions.

1. \[ \begin{align*}
   x - y + z &= 3 \\
   2x + y + z &= 8 \\
   3x + y - z &= 1
\end{align*} \]

2. \[ \begin{align*}
   x - y - z &= 2 \\
   2x + y + z &= 8 \\
   x + y + z &= 6
\end{align*} \]

3. \[ \begin{align*}
   x + y + z &= 1 \\
   2x - y + 2z &= -1 \\
   -x - 3y + z &= 1
\end{align*} \]

4. \[ \begin{align*}
   x - y - z &= 6 \\
   -x + 3y + 2z &= -11 \\
   3x + 2y + z &= 1
\end{align*} \]

5. \[ \begin{align*}
   x + y + z &= 1 \\
   -2x + 2y + 3z &= 20 \\
   2x - 2y - z &= -16
\end{align*} \]

6. \[ \begin{align*}
   x + y + z &= -3 \\
   3x + y - z &= 13 \\
   3x + y - 2z &= 18
\end{align*} \]

7. \[ \begin{align*}
   2x + y - z &= 2 \\
   -x - 3y + z &= -1 \\
   -4x + 3y + z &= -4
\end{align*} \]

8. \[ \begin{align*}
   x + 4y - 6z &= 8 \\
   2x - y + 3z &= -10 \\
   3x - 2y + 3z &= -18
\end{align*} \]

9. \[ \begin{align*}
   3x - y + z &= 5 \\
   x + 3y + 3z &= -6 \\
   x + 4y - 2z &= 12
\end{align*} \]

10. \[ \begin{align*}
   2x - y + 3z &= 2 \\
   x - 2y + 3z &= 1 \\
   4x - y + 5z &= 5
\end{align*} \]

11. \[ \begin{align*}
   x + 2y + z &= 2 \\
   2x + 3y + 3z &= -3 \\
   2x + 3y + 2z &= 2
\end{align*} \]

12. \[ \begin{align*}
   x - 4y - z &= -3 \\
   x + 2y + z &= 5 \\
   3x - 7y - 2z &= -6
\end{align*} \]

13. \[ \begin{align*}
   x + 3y - 2z &= 8 \\
   3x + 2y - 3z &= 15 \\
   4x + 2y + 3z &= -1
\end{align*} \]

14. \[ \begin{align*}
   x + y - z &= 2 \\
   3x + 5y - 2z &= -5 \\
   5x + 4y - 7z &= -7
\end{align*} \]

15. \[ \begin{align*}
   x + y - z &= 2 \\
   x - 2z &= 1 \\
   2x - 3y - z &= 8
\end{align*} \]

16. \[ \begin{align*}
   x + y + z &= 6 \\
   x - 2y &= -7 \\
   4x + 3y + z &= 7
\end{align*} \]

17. \[ \begin{align*}
   x - 3y + 2z &= 1 \\
   16y - 9z &= 5 \\
   4x + 4y - z &= 8
\end{align*} \]

18. \[ \begin{align*}
   x - 4y + 4z &= -1 \\
   y - 3z &= 5 \\
   3x - 4y + 6z &= 1
\end{align*} \]

19. \[ \begin{align*}
   x + 2y - 4z &= 13 \\
   3x + 4y - 2z &= 19 \\
   3x + 2z &= 3
\end{align*} \]

20. \[ \begin{align*}
   x + 2y - z &= 6 \\
   -3x - 2y + 5z &= -12 \\
   x - 2z &= 3
\end{align*} \]
Solve exercises 21 to 36 by choosing a variable to represent each unknown quantity and writing a system of equations.

21. **Number problem.** The sum of three numbers is 16. The largest number is equal to the sum of the other two, and 3 times the smallest number is 1 more than the largest. Find the three numbers.

22. **Number problem.** The sum of three numbers is 24. Twice the smallest number is 2 less than the largest number, and the largest number is equal to the sum of the other two. What are the three numbers?

23. **Coin problem.** A cashier has 25 coins consisting of nickels, dimes, and quarters with a value of $4.90. If the number of dimes is 1 less than twice the number of nickels, how many of each type of coin does she have?

24. **Recreation.** A theater has tickets at $6 for adults, $3.50 for students, and $2.50 for children under 12 years old. A total of 278 tickets were sold for one showing with a total revenue of $1300. If the number of adult tickets sold was 10 less than twice the number of student tickets, how many of each type of ticket were sold for the showing?

25. **Geometry.** The perimeter of a triangle is 19 cm. If the length of the longest side is twice that of the shortest side and 3 cm less than the sum of the lengths of the other two sides, find the lengths of the three sides.

26. **Geometry.** The measure of the largest angle of a triangle is 10° more than the sum of the measures of the other two angles and 10° less than 3 times the measure of the smallest angle. Find the measures of the three angles of the triangle.

27. **Investments.** Jovita divides $17,000 into three investments: a savings account paying 6% annual interest, a bond paying 9%, and a money market fund paying 11%. The annual interest from the three accounts is $1540, and she has three times as much invested in the bond as in the savings account. What amount does she have invested in each account?

28. **Investments.** Adrienne has $6000 invested among a savings account paying 3%, a time deposit paying 4%, and a bond paying 8%. She has $1000 less invested in the bond than in her savings account, and she earned a total of $260 in annual interest. What has she invested in each account?
29. Number problem. The sum of the digits of a three-digit number is 9, and the tens digit of the number is twice the hundreds digit. If the digits are reversed in order, the new number is 99 more than the original number. What is the original number?

30. Number problem. The sum of the digits of a three-digit number is 9. The tens digit is 3 times the hundreds digit. If the digits are reversed in order, the new number is 99 less than the original number. Find the original three-digit number.

31. Motion. Roy, Sally, and Jeff drive a total of 50 mi to work each day. Sally drives twice as far as Roy, and Jeff drives 10 mi farther than Sally. Use a system of three equations in three unknowns to find how far each person drives each day.

32. Consumer affairs. A parking lot has spaces reserved for motorcycles, cars, and vans. There are five more spaces reserved for vans than for motorcycles. There are three times as many car spaces as van and motorcycle spaces combined. If the parking lot has 180 total reserved spaces, how many of each type are there?

The solution process illustrated in this section can be extended to solving systems of more than three variables in a natural fashion. For instance, if four variables are involved, eliminate one variable in the system and then solve the resulting system in three variables as before. Substituting those three values into one of the original equations will provide the value for the remaining variable and the solution for the system.

In exercises 33 and 34, use this procedure to solve the system.

33. \[ \begin{align*} x + 2y + 3z + w &= 0 \\ -x - y - 3z + w &= -2 \\ x - 3y + 2z + 2w &= -11 \\ -x + y - 2z + w &= 1 \end{align*} \]

34. \[ \begin{align*} x + y - 2z - w &= 4 \\ x - y + z + 2w &= 3 \\ 2x + y - z - w &= 7 \\ x - y + 2z + w &= 2 \end{align*} \]

In some systems of equations there are more equations than variables. We can illustrate this situation with a system of three equations in two variables. To solve this type of system, pick any two of the equations and solve this system. Then substitute the solution obtained into the third equation. If a true statement results, the solution used is the solution to the entire system. If a false statement occurs, the system has no solution.

In exercises 35 and 36, use this procedure to solve each system.

35. \[ \begin{align*} x - y &= 5 \\ 2x + 3y &= 20 \\ 4x + 5y &= 38 \end{align*} \]

36. \[ \begin{align*} 3x + 2y &= 6 \\ 5x + 7y &= 35 \\ 7x + 9y &= 8 \end{align*} \]
37. Experiments have shown that cars \((C)\), trucks \((T)\), and buses \((B)\) emit different amounts of air pollutants. In one such experiment, a truck emitted 1.5 pounds (lb) of carbon dioxide (CO\(_2\)) per passenger-mile and 2 grams (g) of nitrogen oxide (NO) per passenger-mile. A car emitted 1.1 lb of CO\(_2\) per passenger-mile and 1.5 g of NO per passenger-mile. A bus emitted 0.4 lb of CO\(_2\) per passenger-mile and 1.8 g of NO per passenger-mile. A total of 85 mi was driven by the three vehicles, and 73.5 lb of CO\(_2\) and 149.5 g of NO were collected. Use the following system of equations to determine the miles driven by each vehicle.

\[
\begin{align*}
T + C + B &= 85.0 \\
1.5T + 1.1C + 0.4B &= 73.5 \\
2T + 1.5C + 1.8B &= 149.5
\end{align*}
\]

38. Experiments have shown that cars \((C)\), trucks \((T)\), and trains \((R)\) emit different amounts of air pollutants. In one such experiment, a truck emitted 0.8 lb of carbon dioxide per passenger-mile and 1 g of nitrogen oxide per passenger-mile. A car emitted 0.7 lb of CO\(_2\) per passenger-mile and 0.9 g of NO per passenger-mile. A train emitted 0.5 lb of CO\(_2\) per passenger-mile and 4 g of NO per passenger-mile. A total of 141 mi was driven by the three vehicles, and 82.7 lb of CO\(_2\) and 424.4 g of NO were collected. Use the following system of equations to determine the miles driven by each vehicle.

\[
\begin{align*}
T + C + R &= 141.0 \\
0.8T + 0.7C + 0.5R &= 82.7 \\
T + 0.9C + 4R &= 424.4
\end{align*}
\]

39. In Chapter 9 you will learn about quadratic functions and their graphs. A quadratic function has the form \(y = ax^2 + bx + c\), in which \(a\), \(b\), and \(c\) are specific numbers and \(a \neq 0\). Three distinct points on the graph are enough to determine the equation.

(a) Suppose that \((1, 5)\), \((2, 10)\), and \((3, 19)\) are on the graph of \(y = ax^2 + bx + c\). Substituting the pair \((1, 5)\) into this equation (that is, let \(x = 1\) and \(y = 5\)) yields \(5 = a + b + c\). Substituting each of the other ordered pairs yields: \(10 = 4a + 2b + c\) and \(19 = 9a + 3b + c\). Solve the resulting system of equations to determine the values of \(a\), \(b\), and \(c\). Then write the equation of the function.

(b) Repeat the work of part (a) using the following three points: \((1, 2)\), \((2, 9)\), and \((3, 22)\).

**Answers**

1. \(\{(1, 2, 4)\}\)  
3. \(\{(-2, 1, 2)\}\)  
5. \(\{(-4, 3, 2)\}\)  
7. Infinite number of solutions  
9. \(\left\{\left(\frac{3}{2}, \frac{1}{2}, \frac{-7}{2}\right)\right\}\)  
11. \(\{(3, 2, -5)\}\)  
13. \(\{(2, 0, -3)\}\)  
15. \(\left\{\left(4, -\frac{1}{2}, \frac{3}{2}\right)\right\}\)  
17. Inconsistent system  
19. \(\left\{\left(\frac{5}{2}, \frac{3}{2}, \frac{-3}{2}\right)\right\}\)  
21. 3, 5, 8  
23. 3 nickels, 5 dimes, 17 quarters  
25. 4 cm, 7 cm, 8 cm  
27. $3000 savings, $9000 bond, $5000 money market  
29. 243  
31. Roy 8 mi, Sally 16 mi, Jeff 26 mi  
33. \((1, 2, -1, -2)\)  
35. \((7, 2)\)  
37. \(T = 20, C = 25, B = 40\)  
39. (a) \(y = 2x^2 - x + 4\); (b) \(y = 3x^2 - 2x + 1\)