A reader in the *Ask Marilyn* column by Marilyn vos Savant posed the following problem:

Assume that the probability of giving birth to a boy is the same as the probability of giving birth to a girl. A woman and a man (unrelated) each have two children. At least one of the woman’s children is a boy, and the man’s first child is a boy. Is the probability that the woman has two boys the same as the probability that the man has two boys?
The lists

\[ 1, 4, 9, 16, 25, 36, 49, 64, \ldots \]

and

\[ 3, 6, 3, 1, 4, 2, 1, 4, \ldots \]

are examples of sequences. The first sequence exhibits a great deal of regularity. You no doubt recognize it as the sequence of perfect squares. Its terms are increasing, and the differences between terms form a striking pattern. You probably do not recognize the second sequence, whose terms do not suggest an obvious pattern. In fact, the second sequence records the results of repeatedly tossing a single die. Sequences, and the related concept of series, are useful tools in almost all areas of mathematics. In this chapter they play roles in the development of several topics: a method of proof called mathematical induction, techniques for counting, and probability.

Preparing for This Chapter
Before getting started on this chapter, review the following concepts:
- Set Notation (Appendix A, Section 1)
- Operations on Polynomials (Appendix A, Section 2)
- Integer Exponents (Appendix A, Section 5)
- Functions (Chapter 1, Section 3)
- Set Operations (Appendix A, Section 8)

Section 6-1 Sequences and Series

In this section we introduce special notation and formulas for representing and generating sequences and sums of sequences.

Sequences
Consider the function \( f \) given by

\[ f(n) = 2n - 1 \quad (1) \]

where the domain of \( f \) is the set of natural numbers \( \mathbb{N} \). Note that

\[ f(1) = 1, f(2) = 3, f(3) = 5, \ldots \]

The function \( f \) is an example of a sequence. A sequence is a function with domain a set of successive integers. However, a sequence is hardly ever represented in
the form of equation (1). A special notation for sequences has evolved, which we describe here.

To start, the range value \( f(n) \) is usually symbolized more compactly with a symbol such as \( a_n \). Thus, in place of equation (1) we write

\[
a_n = 2n - 1
\]

The domain is understood to be the set of natural numbers \( N \) unless stated to the contrary or the context indicates otherwise. The elements in the range are called terms of the sequence: \( a_1 \) is the first term, \( a_2 \) the second term, and \( a_n \) the \( n \)th term, or the general term:

\[
\begin{align*}
a_1 &= 2(1) - 1 = 1 & \text{First term} \\
a_2 &= 2(2) - 1 = 3 & \text{Second term} \\
a_3 &= 2(3) - 1 = 5 & \text{Third term} \\
\vdots & \vdots
\end{align*}
\]

The ordered list of elements

1, 3, 5, \ldots, 2n - 1, \ldots

in which the terms of a sequence are written in their natural order with respect to the domain values, is often informally referred to as a sequence. A sequence is also represented in the abbreviated form \( \{a_n\} \), where a symbol for the \( n \)th term is placed between braces. For example, we can refer to the sequence

1, 3, 5, \ldots, 2n - 1, \ldots

as the sequence \( \{2n - 1\} \).

If the domain of a function is a finite set of successive integers, then the sequence is called a finite sequence. If the domain is an infinite set of successive integers, then the sequence is called an infinite sequence. The sequence \( \{2n - 1\} \) above is an example of an infinite sequence.

Explore/Discuss

1

The sequence \( \{2n - 1\} \) is a function whose domain is the set of natural numbers, and so it may be graphed in the same way as any function whose domain and range are sets of real numbers (see Fig. 1).

**FIGURE 1** Graph of \( \{2n - 1\} \).
Explore/Discuss 1 continued

(A) Explain why the graph of the sequence \( \{2n - 1\} \) is not continuous.

(B) Explain why the points on the graph of \( \{2n - 1\} \) lie on a line. Find an equation for that line.

(C) Graph the sequence \( \left\{ \frac{2n^2 - n + 1}{n} \right\} \). How are the graphs of \( \{2n - 1\} \) and \( \left\{ \frac{2n^2 - n + 1}{n} \right\} \) related?

There are several different ways a graphing utility can be used in the study of sequences. Refer to Explore/Discuss 1. Figure 2(a) shows the sequence \( \{2n - 1\} \) entered as a function in an equation editor. This produces a continuous graph [Fig. 2(b)] that contains the points in the graph of the sequence (Fig. 1). Figure 2(c) shows the points on the graph of the sequence displayed in a table.

In Figure 3(a), sequence commands are used to store the first and second coordinates of the first 10 points on the graph of the sequence \( \{2n - 1\} \) in lists \( L_1 \) and \( L_2 \), respectively. A statistical plot routine is used to graph these points [Fig. 3(b)], and a statistical editor is used to display the points on the graph [Fig. 3(c)].

Most graphing utilities can produce the results shown in Figures 2 and 3. The Texas Instruments TI-83 has a special sequence mode that is very useful for studying sequences. Figure 4(a) shows the sequence \( \{2n - 1\} \) entered in the sequence editor, Figure 4(b) shows the graph of this sequence, and Figure 4(c) displays the points on the graph in a table.
Examining graphs and displaying values are very helpful activities when working with sequences. Consult your manual to see which of the methods illustrated in Figures 2–4 works on your graphing utility.

Some sequences are specified by a recursion formula—that is, a formula that defines each term in terms of one or more preceding terms. The sequence we have chosen to illustrate a recursion formula is a very famous sequence in the history of mathematics called the Fibonacci sequence. It is named after the most celebrated mathematician of the thirteenth century, Leonardo Fibonacci from Italy (1180–1250).

**Fibonacci Sequence**

List the first six terms of the sequence specified by

\[ a_1 = 1 \]
\[ a_2 = 1 \]
\[ a_n = a_{n-1} + a_{n-2} \quad n \geq 3 \]

**Solution**

\[ a_1 = 1 \]
\[ a_2 = 1 \]
\[ a_3 = a_2 + a_1 = 1 + 1 = 2 \]
\[ a_4 = a_3 + a_2 = 2 + 1 = 3 \]
\[ a_5 = a_4 + a_3 = 3 + 2 = 5 \]
\[ a_6 = a_5 + a_4 = 5 + 3 = 8 \]

The formula \( a_n = a_{n-1} + a_{n-2} \) is a recursion formula that can be used to generate the terms of a sequence in terms of preceding terms. Of course, starting terms \( a_1 \) and \( a_2 \) must be provided to use the formula. Recursion formulas are particularly suitable for use with calculators and computers (see Problems 57 and 58 in Exercise 6-1).

**Matched Problem 1**

List the first five terms of the sequence specified by

\[ a_1 = 4 \]
\[ a_n = \frac{1}{2} a_{n-1} \quad n \geq 2 \]
A multiple-choice test question asked for the next term in the sequence:

1, 3, 9, . . .

and gave the following choices:
(A) 16  (B) 19  (C) 27

Which is the correct answer?

Compare the first four terms of the following sequences:

(A) \(a_n = 3^{n-1}\)  (B) \(b_n = 1 + 2(n - 1)^2\)  (C) \(c_n = 8n + \frac{12}{n} - 19\)

Now which of the choices appears to be correct?

Now we consider the reverse problem. That is, can a sequence be defined just by listing the first three or four terms of the sequence? And can we then use these initial terms to find a formula for the \(n\)th term? In general, without other information, the answer to the first question is no. As Explore/Discuss 2 illustrates, many different sequences may start off with the same terms. Simply listing the first three terms, or any other finite number of terms, does not specify a particular sequence. In fact, it can be shown that given any list of \(m\) numbers, there are an infinite number of sequences whose first \(m\) terms agree with these given numbers.

What about the second question? That is, given a few terms, can we find the general formula for at least one sequence whose first few terms agree with the given terms? The answer to this question is a qualified yes. If we can observe a simple pattern in the given terms, then we may be able to construct a general term that will produce the pattern. The next example illustrates this approach.

**Example 2**

**Finding the General Term of a Sequence**

Find the general term of a sequence whose first four terms are

(A) 5, 6, 7, 8, . . .  (B) 2, −4, 8, −16, . . .

**Solutions**

(A) Since these terms are consecutive integers, one solution is \(a_n = n, n \geq 5\). If we want the domain of the sequence to be all natural numbers, then another solution is \(b_n = n + 4\).

(B) Each of these terms can be written as the product of a power of 2 and a power of \(-1\):

\[
2 = (-1)^0 2^1 \\
-4 = (-1)^1 2^2 \\
8 = (-1)^2 2^3 \\
-16 = (-1)^3 2^4
\]

If we choose the domain to be all natural numbers, then a solution is

\[a_n = (-1)^{n-1} 2^n\]
Find the general term of a sequence whose first four terms are

(A) 2, 4, 6, 8, . . .
(B) 1, −1/2, 1/4, −1/8, . . .

In general, there is usually more than one way of representing the $n$th term of a given sequence. This was seen in the solution of Example 2, part A. However, unless stated to the contrary, we assume the domain of the sequence is the set of natural numbers $N$.

The sequence with general term $b_n = \frac{\sqrt{5}}{5} \left( \frac{1 + \sqrt{5}}{2} \right)^n$ is closely related to the Fibonacci sequence. Compute the first 20 terms of both sequences and discuss the relationship. [The first seven values of $b_n$ are shown in Fig. 5(b)].

**Series**

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is called a **series**. If the sequence is finite, the corresponding series is a **finite series**. If the sequence is infinite, the corresponding series is an **infinite series**. For example,

1, 2, 4, 8, 16 \hspace{1cm} \text{Finite sequence}

1 + 2 + 4 + 8 + 16 \hspace{1cm} \text{Finite series}

We restrict our discussion to finite series in this section.

Series are often represented in a compact form called **summation notation** using the symbol $\sum$, which is a stylized version of the Greek letter sigma. Consider the following examples:

\[
\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4 \\
\sum_{k=3}^{7} b_k = b_3 + b_4 + b_5 + b_6 + b_7 \\
\sum_{k=0}^{n} c_k = c_0 + c_1 + c_2 + \cdots + c_n \\
\text{Domain is the set of integers } k \text{ satisfying } 0 \leq k \leq n.
\]
The terms on the right are obtained from the expression on the left by successively replacing the summing index $k$ with integers, starting with the first number indicated below $\sum$ and ending with the number that appears above $\sum$. Thus, for example, if we are given the sequence
\[
\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}
\]
the corresponding series is
\[
\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}
\]

**Example 3**

**Writing the Terms of a Series**

Write without summation notation: $\sum_{k=1}^{5} \frac{k-1}{k}$

**Solution**

\[
\sum_{k=1}^{5} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} + \frac{4-1}{4} + \frac{5-1}{5} = 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}
\]

**Matched Problem 3**

Write without summation notation: $\sum_{k=0}^{5} \frac{(-1)^k}{2k + 1}$

If the terms of a series are alternately positive and negative, it is called an alternating series. Example 4 deals with the representation of such a series.

**Example 4**

**Writing a Series in Summation Notation**

Write the following series using summation notation:

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}
\]

(A) Start the summing index at $k = 1$.

(B) Start the summing index at $k = 0$.

**Solutions**

(A) $(-1)^{k-1}$ provides the alternation of sign, and $1/k$ provides the other part of each term. Thus, we can write

\[
\sum_{k=1}^{6} \frac{(-1)^{k-1}}{k}
\]

as can be easily checked.
(B) \((-1)^k\) provides the alternation of sign, and \(1/(k+1)\) provides the other part of each term. Thus, we write

\[
\sum_{k=0}^{s} \frac{(-1)^k}{k+1}
\]

as can be checked.

**MATCHED PROBLEM 4**

Write the following series using summation notation:

\[
1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81}
\]

(A) Start with \(k = 1\).  
(B) Start with \(k = 0\).

(A) Find the smallest number of terms of the infinite series

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots
\]

that, when added together, give a number greater than 3.

(B) Find the smallest number of terms of the infinite series

\[
\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots
\]

that, when added together, give a number greater than 0.99. Greater than 0.999. Can the sum ever exceed 1? Explain.

**Answers to Matched Problems**

1. 4, 2, 1, \(\frac{1}{2}, \frac{1}{3}\) 2. (A) \(a_n = 2n\)  
   (B) \(a_n = (-1)^{n-1} \left(\frac{1}{2}\right)^{n-1}\) 3. \(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}\)

4. (A) \(\sum_{k=1}^{5} (-1)^{k-1} \left(\frac{2}{3}\right)^{k-1}\)  
   (B) \(\sum_{k=0}^{4} (-1)^k \left(\frac{2}{3}\right)^k\)

5. \(a_n = (-2)^{n+1}\) 6. \(a_n = \frac{(-1)^{n+1}}{n^2}\)

7. Write the eighth term in the sequence in Problem 1.
8. Write the tenth term in the sequence in Problem 2.
9. Write the one-hundredth term in the sequence in Problem 3.
10. Write the two-hundredth term in the sequence in Problem 4.

**EXERCISE 6-1**

Write the first four terms for each sequence in Problems 1–6.

1. \(a_n = n - 2\) 2. \(a_n = n + 3\)
3. \(a_n = \frac{n-1}{n+1}\) 4. \(a_n = \left(1 + \frac{1}{n}\right)^n\)

5. \(a_n = (-2)^{n+1}\) 6. \(a_n = \frac{(-1)^{n+1}}{n^2}\)
In Problems 11–16, write each series in expanded form without summation notation.

11. \( \sum_{k=1}^{n} k \)
12. \( \sum_{k=1}^{n} k^2 \)
13. \( \sum_{k=1}^{n} \frac{1}{10^k} \)
14. \( \sum_{k=1}^{n} \left( \frac{1}{3} \right)^k \)
15. \( \sum_{k=1}^{n} (-1)^k \)
16. \( \sum_{k=1}^{n} (-1)^{k+1}k \)

Write the first five terms of each sequence in Problems 17–26.

17. \( a_n = (-1)^{n+1}n^2 \)
18. \( a_n = (-1)^{n+1} \left( \frac{1}{2^n} \right) \)
19. \( a_n = \frac{1}{3} \left( 1 - \frac{1}{10^n} \right) \)
20. \( a_n = n[1 - (-1)^n] \)
21. \( a_1 = 7; a_n = a_{n-1} - 4, n \geq 2 \)
22. \( a_1 = 2; a_n = 2a_{n-1} + 1, n \geq 2 \)
23. \( a_1 = 4; a_n = \frac{1}{2}a_{n-2}, n \geq 2 \)
24. \( a_1 = 1, a_2 = 2; a_n = a_{n-1} + a_{n-2}, n \geq 3 \)
25. \( a_1 = 1; a_n = a_{n-1} + a_{n-2} \)
26. \( a_1 = -1; a_n = -a_{n-1} \)

In Problems 27–38, find the general term of a sequence whose first four terms are given.

27. \( 4, 5, 6, 7, \ldots \)
28. \( -2, -1, 0, 1, \ldots \)
29. \( 3, 6, 9, 12, \ldots \)
30. \( -2, -4, -6, -8, \ldots \)
31. \( \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{2}, \ldots \)
32. \( \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \ldots \)
33. \( 1, -1, 1, -1, \ldots \)
34. \( 1, -2, 3, -4, \ldots \)
35. \( -2, 4, -8, 16, \ldots \)
36. \( 1, -3, 5, -7, \ldots \)
37. \( \sqrt{x}, \sqrt[3]{x}, \sqrt[4]{x}, \sqrt[5]{x}, \ldots \)
38. \( x, -x^3, x^5, -x^7, \ldots \)

In Problems 39–42, use a graphing utility to graph the first 20 terms of each sequence.

39. \( a_n = 1/n \)
40. \( a_n = 2 + \pi n \)
41. \( a_n = (-0.9)^n \)
42. \( a_1 = -1, a_n = \frac{3}{5}a_{n-1} + \frac{1}{2} \)

In Problems 43–48, write each series in expanded form without summation notation.

43. \( \sum_{k=1}^{n} \frac{(-2)^{k+1}}{k} \)
44. \( \sum_{k=1}^{n} (-1)^{k+1}(2k - 1)^3 \)
45. \( \sum_{k=1}^{n} \frac{1}{k}x^{k+1} \)
46. \( \sum_{k=1}^{n} x^{k-1} \)
47. \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \)
48. \( \sum_{k=0}^{n} \frac{(-1)^{k+1}2k + 1}{2k + 1} \)

In Problems 49–56, write each series using summation notation with the summing index \( k \) starting at \( k = 1 \).

49. \( 2 + 2^2 + 3^2 + 4^2 \)
50. \( 2 + 3 + 4 + 5 + 6 \)
51. \( \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} \)
52. \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \)
53. \( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \)
54. \( \frac{3}{2} + \frac{1}{3} + \ldots + \frac{n + 1}{n} \)
55. \( 1 - 4 + 9 - \ldots - (-1)^{n+1}n^2 \)
56. \( \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots - (-1)^{n+1} \frac{1}{2^n} \)

The sequence
\[ a_n = \frac{a_{n-1}^2 + M}{2a_{n-1}} \quad n \geq 2, \quad M \text{ a positive real number} \]
can be used to find \( \sqrt{M} \) to any decimal-place accuracy desired. To start the sequence, choose \( a_1 \) arbitrarily from the positive real numbers. Problems 57 and 58 are related to this sequence.

57. (A) Find the first four terms of the sequence
\[ a_1 = 3 \quad a_n = \frac{a_{n-1}^2 + 2}{2a_{n-1}} \quad n \geq 2 \]
(B) Compare the terms with \( \sqrt{2} \) from a calculator.
(C) Repeat parts A and B letting \( a_1 \) be any other positive number, say 1.

58. (A) Find the first four terms of the sequence
\[ a_1 = 2 \quad a_n = \frac{a_{n-1}^2 + 5}{2a_{n-1}} \quad n \geq 2 \]
(B) Find \( \sqrt{5} \) with a calculator, and compare with the results of part A.
(C) Repeat parts A and B letting \( a_1 \) be any other positive number, say 3.

59. Let \( \{a_n\} \) denote the Fibonacci sequence and let \( \{b_n\} \) denote the sequence defined by \( b_1 = 1, b_2 = 3, b_n = b_{n-1} + b_{n-2} \) for \( n \geq 3 \). Compute 10 terms of the sequence \( \{c_n\} \), where \( c_n = b_n/a_n \). Describe the terms of \( \{c_n\} \) for large values of \( n \).

60. Define sequences \( \{u_n\} \) and \( \{v_n\} \) by \( u_1 = 1, v_1 = 0, u_n = u_{n-1} + v_{n-1} \) and \( v_n = u_{n-1} \) for \( n \geq 2 \). Find the first 10 terms of each sequence, and explain their relationship to the Fibonacci sequence.
In calculus, it can be shown that

\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \]

where the larger \( n \) is, the better the approximation. Problems 61 and 62 refer to this series. Note that \( n! \), read “\( n \) factorial,” is defined by \( 0! = 1 \) and \( n! = 1 \cdot 2 \cdot 3 \cdots n \) for \( n \in \mathbb{N} \).

61. Approximate \( e^{0.2} \) using the first five terms of the series. Compare this approximation with your calculator evaluation of \( e^{0.2} \).

62. Approximate \( e^{-0.5} \) using the first five terms of the series. Compare this approximation with your calculator evaluation of \( e^{-0.5} \).

63. Show that \( \sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k \)

64. Show that \( \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \)

Section 6-2 Mathematical Induction

- Introduction
- Mathematical Induction
- Additional Examples of Mathematical Induction
- Three Famous Problems

Introduction

In common usage, the word “induction” means the generalization from particular cases or facts. The ability to formulate general hypotheses from a limited number of facts is a distinguishing characteristic of a creative mathematician. The creative process does not stop here, however. These hypotheses must then be proved or disproved. In mathematics, a special method of proof called mathematical induction ranks among the most important basic tools in a mathematician’s toolbox. In this section, mathematical induction will be used to prove a variety of mathematical statements, some new and some that up to now we have just assumed to be true.

We illustrate the process of formulating hypotheses by an example. Suppose we are interested in the sum of the first \( n \) consecutive odd integers, where \( n \) is a positive integer. We begin by writing the sums for the first few values of \( n \) to see if we can observe a pattern:

\[
\begin{align*}
1 & = 1 \quad n = 1 \\
1 + 3 & = 4 \quad n = 2 \\
1 + 3 + 5 & = 9 \quad n = 3 \\
1 + 3 + 5 + 7 & = 16 \quad n = 4 \\
1 + 3 + 5 + 7 + 9 & = 25 \quad n = 5 \\
\end{align*}
\]

Is there any pattern to the sums 1, 4, 9, 16, and 25? You no doubt observed that each is a perfect square and, in fact, each is the square of the number of terms in the sum. Thus, the following conjecture seems reasonable:

**Conjecture** \( P_n \): For each positive integer \( n \),

\[ 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \]

That is, the sum of the first \( n \) odd integers is \( n^2 \) for each positive integer \( n \).